Left-symmetric algebras

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An algebra $\mathfrak{A}$ (where the product is denoted by $ab = \mathrm{L}_a(b) = \mathrm{R}_b(a)$) is called *left-symmetric* if for all $a, b, c \in \mathfrak{A}$, we have

$$a(bc) - (ab)c = b(ac) - (ba)c.$$

The bracket $[a, b] = ab - ba$ is the bracket of a Lie algebra which we shall call *subordinate* to $\mathfrak{A}$, and the map $(a, b) \mapsto a + ab$ determines an affine representation of the Lie algebra $\mathfrak{A}$ on the affine space $\mathfrak{A}$.

On the other hand, if $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field $\mathbb{K}$ of characteristic 0, we want to recognize the left-symmetric algebras for which $\mathfrak{g}$ is the subordinate Lie algebra. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and if G is simply connected and $\mathfrak{g}$ its Lie algebra of left-invariant vector fields, this problem can be rephrased as any one of the following two problems:

Problem 1: Equip G with locally flat left-invariant affine connection. The product of two left-invariant vector fields is then the covariant derivative of the second one by the first.

Problem 2: Study the affine representations of G on an affine space of the same dimension as G, such that G has an open orbit. In fact, the natural flat affine connection on this orbit can be pulled back to the group G after choice of a base point in the orbit.

1 The radical of a left-symmetric algebra $\mathfrak{A}$

Proposition 1 *For all $u \in \mathfrak{A}$ the three following assertions are equivalent:*

(1) *The left-ideal generated by u in $\mathfrak{A}$ is contained in the kernel of the linear form $a \mapsto \mathrm{tr}(\mathrm{R}_a)$.*

(2) *The left-ideal generated by u is contained in the set of $a \in \mathfrak{A}$ such that R_a is nilpotent.*

(3) *For all $a \in \mathfrak{A}$ we have $\det(I + \mathrm{R}_a) = \det(I + \mathrm{R}_{a+u})$.*

The set of $u \in \mathfrak{A}$ that satisfies these properties is a left-ideal $\mathfrak{R}(\mathfrak{A})$, called the *radical* of $\mathfrak{A}$. This radical is contained in the orthogonal space of $\mathfrak{A}$ for the symmetric bilinear form

$$(a, b) \mapsto \mathrm{tr}(\mathrm{R}_{ab}) = \mathrm{tr}(\mathrm{R}_a \mathrm{R}_b),$$

and I conjecture that these are equal.[1)]

Here are the other results concerning the radical:

Proposition 2 *If $\mathfrak{R}(\mathfrak{A}) = \mathfrak{A}$, then the subordinate Lie algebra is solvable.*

Corollary *If the Lie algebra $\mathfrak{g}$ equals its derived algebra $[\mathfrak{g}, \mathfrak{g}]$, then it cannot be a subordinate Lie algebra for a left-symmetric algebra.*

Proposition 3 *If the affine representation defined by the $(a, b) \mapsto a + ab$ is similar to linear semisimple representation, the subordinate Lie algebra is reductive and $\mathfrak{R}(\mathfrak{A}) = \mathbf{0}$.*

2 Convex homogeneous cones

If in a real affine space there exists an open convex cone Ω on which the group of affine transformations preserving Ω acts transitively, then one can find within this group a subgroup G that acts simply transitively on Ω. From what was explained in the beginning, this implies that there exists a structure of a left-symmetric algebra on the Lie algebra $\mathfrak{g}$ of G. It satisfies the following property that characterizes those left-symmetric algebras that can be obtained in this way: there exist linear forms s on $\mathfrak{g}$ (for example $s(a) = \mathrm{tr}(\mathrm{L}_a)$) such that the bilinear form $(a, b) \mapsto s(ab)$ is symmetric and positive definite. Moreover, the radical is zero and the Lie algebra $\mathfrak{g}$ is solvable.

We now want to study other left-symmetric algebras with zero radical, by dropping the property of positivity that brings us back to algebras of the previous type. We are certain to obtain among these algebras reductive Lie algebras $\mathfrak{g}$ (but not semisimple ones, because of the corollary to Proposition 2), for Proposition 3 gives us a sufficient condition for the radical to be zero in this case.

3 Reductive subordinate Lie algebras

Among the left-symmetric algebras over $\mathbb{C}$ whose subordinate Lie algebra is reductive, there are three fundamental types from which all others can be con-

[1)]Translator's note: This conjecture turns out to be false, see [2, p. 20].

structed. Here are the properties characterizing those types:

Type I: $[\mathfrak{A}, \mathfrak{A}] = \mathbf{0}$ (so these are the associative and commutative algebras).

Type II: The left-symmetric algebras $\mathfrak{S}$ generated by $[\mathfrak{A}, \mathfrak{A}]$ is a simple associative algebra different from $\mathfrak{A}$, and the map $\mathfrak{S} \times \mathfrak{A} \to \mathfrak{A}$ makes $\mathfrak{A}$ into an $\mathfrak{S}$-module.

Type III: The subalgebra $[\mathfrak{A}, \mathfrak{A}]$ equals $\mathfrak{A}$.

Aside from a countable family of special cases, the algebras of type II can be constructed from an algebra $\mathfrak{B}$ of type I by defining on the vector space $\mathfrak{A} = \mathrm{End}(\mathfrak{B}) \oplus \mathfrak{B}$ the (non-associative) product

$$(f, x)(g, y) = (fg + [\mathrm{L}_x, g], f(y) + g(x) + xy),$$

where $f, g \in \mathrm{End}(\mathfrak{B})$, $x, y \in \mathfrak{B}$. The subordinate Lie algebra is the direct product $\mathrm{End}(\mathfrak{B}) \times \mathfrak{B}$.

Aside from these algebras of endomorphisms of a vector space, type III also contains much more complicated irreducible algebras. Here is one example: let V_4, V_3, V_2 be vector spaces of dimensions 4, 3, and 2, respectively, and let $\mathfrak{sp}(V_4)$ be the subalgebra of $\mathfrak{sl}(V_4)$ that fixes an element of $\bigwedge^2 V_4$ of maximal rank, and let $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\mathfrak{z}_3$, $\mathfrak{z}_4$ be the Lie algebras of homotheties of the four vector spaces V_4, $V_4 \otimes V_3$, $V_3 \otimes V_3 2$ and $\mathfrak{sl}(V_2)$. The reductive Lie algebras

$$\mathfrak{g} = \mathfrak{sp}(V_4) \times \mathfrak{sl}(V_3) \times \mathfrak{sl}(V_2) \times \mathfrak{z}_1 \times \mathfrak{z}_2 \times \mathfrak{z}_3 \times \mathfrak{z}_4$$

has a natural representation on the vector space V that is the direct sum of four preceding vector spaces. Since $\mathfrak{g}$ and V have the same dimension 25 and this representation has an open orbit, it follows that there exists a left-symmetric algebra structure on $\mathfrak{g}$.

In order to allow the reduction to the three types above, it is necessary to utilize the methods described in the following.

4 Associative kernels and transformations of the algebras

The *associative kernel* of a left-symmetric algebra $\mathfrak{A}$ is the associative subalgebra $\mathfrak{k}(\mathfrak{A})$ formed by those $u \in \mathfrak{A}$ satisfying

$$a(bu) = (ab)u$$

for all $a, b \in \mathfrak{A}$.[2)] It contains the center $\mathfrak{z}$ of the subordinate Lie algebra $\mathfrak{g}$ because of the following identity:

$$a(bu) - (ab)u = [u, ab] - [u, a]b - a[u, b].$$

The associative kernel allows to construct out of $\mathfrak{A}$ other left-symmetric algebras with the same subordinate Lie algebra as $\mathfrak{A}$. In fact, if h is an endomorphism of the subordinate Lie algebra such that $I - h$ is bijective an $h(\mathfrak{A}) \subset \mathfrak{k}(\mathfrak{A})$, we can associate to the pair $(\mathfrak{A}, h)$ a new algebra $\mathfrak{A}'$, called a *transform* of $\mathfrak{A}$, on which the product (written $a \times b = \mathrm{L}'_a(b) = \mathrm{R}'_b(a)$) is defined by either one of the following formulae (for the second one set $\varphi = (I - h)^{-1}$):

$$\mathrm{L}'_a = (I - h)^{-1}(\mathrm{L}_a - \mathrm{R}_{h(a)})(I - h), \tag{1}$$

$$\varphi(a) \times \varphi(b) = \varphi(ab + [(\varphi - I)(a), b]). \tag{2}$$

The most interesting case is the one where h (hence also $\varphi - I$) vanishes on the derived Lie algebra, which implies that $h(\mathfrak{A})$ is an abelian subalgebra (note that $h(\mathfrak{A}) = (\varphi - I)(\mathfrak{A})$). In this case, the transformation is said to be *elementary*. The effects of such a transformation can be understood well (Proposition 4 below) and the elementary transformations form a group (Proposition 5).

Proposition 4 *If h vanishes on $[\mathfrak{A}, \mathfrak{A}]$ and if $\mathfrak{C}$ denotes the centralizer of $h(\mathfrak{A})$, then:*

$$\mathfrak{k}(\mathfrak{A}') \cap \mathfrak{C} = \mathfrak{k}(\mathfrak{A}) \cap \mathfrak{C},$$
$$\mathfrak{R}(\mathfrak{A}') = \varphi(\mathfrak{R}(\mathfrak{A})).$$

Proposition 5 *Let $\mathfrak{a}$ be an abelian subalgebra of a Lie algebra $\mathfrak{g}$, and let $\mathbf{GL}(\mathfrak{g}, \mathfrak{a})$ be the group of bijective endomorphisms φ of $\mathfrak{g}$ such that $(\varphi - I)([\mathfrak{g}, \mathfrak{g}]) = \mathbf{0}$ and $(\varphi - I)(\mathfrak{g}) \subset \mathfrak{a}$. The group $\mathbf{GL}(\mathfrak{g}, \mathfrak{a})$ acts on the set of left-symmetric algebras to which the Lie algebra $\mathfrak{g}$ is subordinate and for which the associative kernel is contained in $\mathfrak{a}$. The action of this group is defined by formula (2) above. If $\mathfrak{a} \subset \mathfrak{z}$ (the center of $\mathfrak{g}$), the orbits of this group are made up of isomorphic algebras (trivial case).*

The elementary transformations often allow to solve the following problem: if $\mathfrak{A}$ is a left-symmetric algebra over $\mathbb{R}$ or $\mathbb{C}$, find in a neighborhood of $\mathfrak{A}$ those left-symmetric algebras that have the same subordinate Lie algebra. If $\mathfrak{k}(\mathfrak{A}) \neq \mathfrak{z}$, then one finds Lie algebras not isomorphic to $\mathfrak{A}$. If $\mathfrak{A}$ is a semisimple associative algebra, all algebras found in a neighborhood of $\mathfrak{A}$ can be derived by an elementary transformation from $\mathfrak{A}$.

[2)] Translator's note: That is, $\mathfrak{k}(\mathfrak{A}) = \{u \in \mathfrak{A} \mid [\mathrm{L}_a, \mathrm{R}_u] = 0\}$.

References

[1] È.B. Vinberg: *Convex homogeneous cones*. Translations of the Moscow Mathematical Society **12**, 1963 340-403.

[2] J. Hélmstetter, *Radical d'une algèbre symétrique à gauche*, Annales de l'Institut Fourier **29**, 1979, 17-35.

Original: *Algèbres symétriques à gauche*, Comptes Rendus Mathématique, Académie des Sciences, 272, 1971, 1088-1091

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