On the exponential function

By CHARLES HERMITE

I. Given an arbitrary number of numerical quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$, we know that we can approximate them simultaneously by fractions with the same denominator, such that we have

$$\alpha_1 = \frac{A_1}{A} + \frac{\delta_1}{A\sqrt[n]{A}},$$
$$\alpha_2 = \frac{A_2}{A} + \frac{\delta_2}{A\sqrt[n]{A}},$$
$$\vdots$$
$$\alpha_n = \frac{A_n}{A} + \frac{\delta_n}{A\sqrt[n]{A}},$$

where $\delta_1, \delta_2, \ldots, \delta_n$ do not exceed a bound that depends only on n. As we shall see, this is the extension of a mode of approximation resulting from the theory of continued fractions, which correspond to the simplest case of $n = 1$. Now we can propose a generalization related to the algebraic theory of continued fractions, by looking for expressions approximating n functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ by rational fractions $\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \ldots, \frac{\Phi_n(x)}{\Phi(x)}$ in such a way that the expansion into a power series in the variable x coincide up to a certain power x^M. We first consider a result that suggests itself immediately. Suppose the functions $\varphi_1(x)$, $\varphi_2(x)$, …, $\varphi_n(x)$ can all be expanded into a series of the form $\alpha + \beta x + \gamma x^2 + \ldots$ and let

$$\Phi(x) = Ax^m + Bx^{m-1} + \ldots + Kx + L.$$

In general, the coefficients $A, B, \ldots, L$ can be arranged in such a way that in the n products $\varphi_i(x)\Phi(x)$ the terms in

$$x^M, x^{M-1}, \ldots, x^{M-\mu_i+1},$$

vanish, where μ_i is an arbitrary integer. This way, we pose a number of homogeneous equations with lowest degree precisely μ_i, and we have

$$\varphi_i(x)\Phi(x) = \Phi_i(x) + \varepsilon_1 x^{M+1} + \varepsilon_2 x^{M+2} + \ldots,$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are constants and $\Phi_i(x)$ is a polynomial with integer coefficients of degree $M - \mu_i$. Now this relation yields

$$\varphi_i(x) = \frac{\Phi_i(x)}{\Phi(x)} + \frac{\varepsilon_1 x^{M+1} + \varepsilon_2 x^{M+2} + \ldots}{\Phi(x)},$$

and we see that the series expansion of the rational fraction and the function are in fact the same up to the terms in x^M, and, since the total number of posed equations is $\mu_1 + \mu_2 + \ldots + \mu_n$, it is sufficient to impose the sole condition

$$\mu_1 + \mu_2 + \ldots + \mu_n = m,$$

where the integers μ_i remain arbitrary up to here. It is this simple observation that serves as the departure point for my study of the exponential function, where I propose to apply it to the functions $\varphi_1(x) = \mathrm{e}^{ax}$, $\varphi_2(x) = \mathrm{e}^{bx}$, ..., $\varphi_n(x) = \mathrm{e}^{hx}$.

II. For short, let $M - m = \mu$. With the constants $a, b, \ldots, h$ I compose the polynomial

$$F(z) = z^{\mu}(z-a)^{\mu_1}(z-b)^{\mu_2}\cdots(z-h)^{\mu_n}$$

of degree $\mu + \mu_1 + \ldots + \mu_n = M$, and I investigate the n integrals defined by

$$\int_0^a \mathrm{e}^{-zx} F(z)\mathrm{d}z, \int_0^b \mathrm{e}^{-zx} F(z)\mathrm{d}z, \ \ldots, \int_0^h \mathrm{e}^{-zx} F(z)\mathrm{d}z,$$

which are easily obtained in an explicit form. In fact, by setting

$$\mathcal{F}(z) = \frac{F(z)}{x} + \frac{F'(z)}{x^2} + \ldots + \frac{F^{(M)}(z)}{x^{M+1}}$$

we obtain

$$\int \mathrm{e}^{-zx} F(z)\mathrm{d}z = \mathrm{e}^{-zx}\mathcal{F}(z)$$

and consequently,

$$\int_0^a \mathrm{e}^{-zx} F(z)\mathrm{d}z = \mathcal{F}(0) - \mathrm{e}^{ax}\mathcal{F}(a), \int_0^b \mathrm{e}^{-zx} F(z)\mathrm{d}z = \mathcal{F}(0) - \mathrm{e}^{bx}\mathcal{F}(b), \cdots$$

Now the expression for $\mathcal{F}(z)$ follows immediately, in the form of polynomials ordered by ascending powers of $\frac{1}{x}$, the quantities $\mathcal{F}(0), \mathcal{F}(a), \mathcal{F}(b), \ldots$ and if we observe that

$$F(0) = 0, F'(0) = 0, \ldots, F^{(\mu-1)}(0) = 0$$

holds, then successively

$$F(a) = 0, F'(a) = 0, \ldots, F^{(\mu_1-1)} = 0,$$
$$F(b) = 0, F'(b) = 0, \ldots, F^{(\mu_2-1)} = 0,$$
$$\ldots.$$

We conclude the following:

$$\mathcal{F}(0) = \frac{\Phi(x)}{x^{M+1}}, \mathcal{F}(a) = \frac{\Phi_1(x)}{x^{M+1}}, \ldots, \mathcal{F}(h) = \frac{\Phi_n(x)}{x^{M_1+1}},$$

where the integer polynomial $\Phi(x)$ is of degree $M - \mu = m$, and the other $\Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x)$ are of degrees $M - \mu_1, M - \mu_2, \ldots, M - \mu_n$. Having established this, we write

$$e^{ax}\Phi(x) - \Phi_1(x) = x^{M+1}e^{ax}\int_0^a e^{-zx}F(z)\mathrm{d}z,$$
$$e^{bx}\Phi(x) - \Phi_2(x) = x^{M+1}e^{bx}\int_0^b e^{-zx}F(z)\mathrm{d}z,$$
$$\ldots$$
$$e^{hx}\Phi(x) - \Phi_n(x) = x^{M+1}e^{hx}\int_0^h e^{-zx}F(z)\mathrm{d}z,$$

where by expansion of the definite integrals into power series of the form $\alpha + \beta x + \gamma x^2 + \ldots$, we see that the preceding conditions from the definition of a new mode of approximation of functions are completely satisfied. We have thus obtained, in full generality, the system of rational fractions $\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \ldots, \frac{\Phi_n(x)}{\Phi(x)}$ of the same denominator to represent the functions $e^{ax}, e^{bx}, \ldots, e^{hx}$ in terms up to the order x^{M+1}.

III. Let $n = 1$ and assume further $\mu = \mu_1 = m$, which gives $M = 2m$ and $F(z) = z^m(z-1)^m$. The derivatives of $F(z)$ at $z = 0$ can immediately be taken from the expansion by the binomial formula

$$F(z) = z^{2m} - \frac{m}{1}z^{2m-1} + \frac{m(m-1)}{1\cdot 2}z^{2m-2} - \ldots + (-1)^m z^m,$$

and we obtain

$$\frac{F^{(2m-k)}(0)}{1\cdot 2\cdot 3\cdots(2m-k)} = (-1)^k\frac{m(m-1)\cdots(m-k+1)}{1\cdot 2\cdot 3\cdots k},$$

from which immediately

$$\frac{\Phi(x)}{1\cdot 2\cdot 3\cdots m} = 3m(3m-1)\cdots(m+1) - (2m-1)(2m-2)\cdots(m+1)\frac{m}{1}x$$
$$+ (2m-2)(2m-3)\cdots(m+1)\frac{m(m-1)}{1\cdot 2}x^2 - \ldots + (-1)^m x^m$$

follows.

Secondly, to have the values of the derivatives under the assumption $z = 1$, we set $z = 1 + h$ to expand by consecutive powers of h in the polynomial $F(1 + h) = h^m(h + 1)^m$. Now that the previously obtained coefficients are recovered, up to sign, we see that we have

$$\Phi_1(x) = \Phi(-x).$$

These results lead us to introduce, instead of $\Phi(x)$ and $\Phi_1(x)$, the polynomials $\Pi(x) = \frac{\Phi(x)}{1\cdot 2\cdot 3\cdots m}$, $\Pi_1(x) = \frac{\Phi(x)}{1\cdot 2\cdot 3\cdots m}$, whose coefficients are integers. We thus have

$$\begin{aligned} e^x \Pi(x) - \Pi_1(x) &= \frac{x^{2m+1}}{1\cdot 2\cdot 3\cdots m} e^x \int_0^1 e^{-zx} z^m (z-1)^m \mathrm{d}z \\ &= (-1)^m \frac{x^{2m+1}}{1\cdot 2\cdot 3\cdots m} \int_0^1 e^{x(1-z)} z^m (z-1)^m \mathrm{d}z, \end{aligned}$$

and this shows that the first term can become the smallest of the whole given quantity, for sufficiently large values of m. We know indeed that the factor $\frac{x^{2m+1}}{1\cdot 2\cdot 3\cdots m}$ has limit 0, and it is the same for the integral, as the quantity $z^m(10z)^m$ is always smaller than its maximum $(\frac{1}{2})^m$ which decreases indefinitely for growing m. It follows that when assuming x to be integer, the exponential e^x cannot have a commensurable value, for if $e^x = \frac{b}{a}$, we achieve, after getting rid of the denominator, that

$$b\Pi(x) - a\Pi_1(x) = (-1)^m \frac{a x^{2m+1}}{1\cdot 2\cdot 3\cdots m} \int_0^1 e^{x(1-z)} z^m (1-z)^m \mathrm{d}z,$$

whose second term can be become smaller than any given value without ever vanishing, whereas the first term is an integer number. Lambert, to whom we owe this proposition, as well as the only proof to date of the irrationality of the ratio of the circumference to the diameter and its square,derived these important results from the continued fraction

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \ddots}}}$$

to which we will return later. Leaving aside the ratio of the circumference and the diameter, I will now attempt to progress further with regard to the number e, by establishing the impossibility of a relation of the form

$$N + e^a N_1 + e^b N_2 + \ldots + e^h N_n = 0,$$

where $a, b, \ldots, h$ and the coefficients $N, N_1, \ldots, N_n$ are integer numbers.

IV. To this end, I consider among the systems of rational fractions $\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \dots, \frac{\Phi_n(x)}{\Phi(x)}$ those which are obtained by assuming $\mu = \mu_1 = \ldots = \mu_n$, which gives

$$m = n\mu, \quad M = (n+1)\mu, \quad F(z) = f^{\mu}(z),$$

where $f(z) = z(z-a)(z-b)\cdots(z-h)$. As usual, let then

$$\Pi(x) = \frac{\Phi(x)}{1\cdot 2\cdot 3\cdots\mu}, \Pi_1(x) = \frac{\Phi_1(x)}{1\cdot 2\cdot 3\cdots\mu}, \dots, \Pi_n(x) = \frac{\Phi_n(x)}{1\cdot 2\cdot 3\cdots\mu}$$

denote the polynomials having as coefficients integer numbers and leading to the following relations:

$$\begin{aligned} e^{ax}\Pi(x) - \Pi_1(x) &= \varepsilon_1, \\ e^{bx}\Pi(x) - \Pi_2(x) &= \varepsilon_2, \\ &\vdots \\ e^{hx}\Pi(x) - \Pi_n(x) &= \varepsilon_n \end{aligned} \tag{A}$$

where we write for short

$$\begin{aligned} \varepsilon_1 &= \frac{x^{M+1}e^{ax}}{1\cdot 2\cdot 3\cdots\mu}\int_0^a e^{-zx}F(z)dz = \int_0^a e^{x(a-z)}\frac{f^{\mu}(z)x^{(n+1)\mu+1}}{1\cdot 2\cdot 3\cdots\mu}dz, \\ \varepsilon_2 &= \frac{x^{M+1}e^{bx}}{1\cdot 2\cdot 3\cdots\mu}\int_0^b e^{-zx}F(z)dz = \int_0^b e^{x(b-z)}\frac{f^{\mu}(z)x^{(n+1)\mu+1}}{1\cdot 2\cdot 3\cdots\mu}dz, \\ &\vdots \end{aligned}$$

With this in place, I observe first that for a sufficiently large value of μ, the values of $\varepsilon_1, \varepsilon_2, \ldots$ become smaller than any given quantity. For, since the polynomial $f(z)$ does not surpass a certain bound λ in the interval through which the variable runs, the factor $\frac{f^{\mu}(z)x^{(n+1)\mu+1}}{1\cdot 2\cdot 3\cdots\mu}$ which is multiplied with the exponential under the integration sign, is always less than the quantity $\frac{(\lambda x^{n+1})^{\mu}x}{1\cdot 2\cdot 3\cdots\mu}$ which has 0 as limit. Now I assume $x = 1$ in (A), and denote by P_i the value corresponding to $\Pi_i(x)$ which is an integer number under the hypothesis on $a, b, \ldots, h$, they become

$$\begin{aligned} e^{a}P - P_i &= \varepsilon_1, \\ e^{b}P - P_i &= \varepsilon_2, \\ &\vdots \\ e^{h}P - P_i &= \varepsilon_n, \end{aligned}$$

and the assumed relation

$$N + e^a N_1 + e^b N_2 + \ldots + e^h N_n = 0$$

easily gives the following:

$$NP + N_1 P_1 + \ldots + N_n P_n = -(N_1 \varepsilon_1 + N_2 \varepsilon_2 + \ldots + N_n \varepsilon_n),$$

whose first term is essentially integer, the second one, when expressed relative to $\varepsilon_1, \varepsilon_2, \ldots$, can become smaller than any given quantity for growing μ. We thus necessarily have that from a certain value of μ on, for all larger values,

$$NP + N_1 P_1 + \ldots + N_n P_n = 0.$$

Assume therefore that, while μ becomes successively $\mu + 1$, $\mu + 2, \ldots, \mu + n$, P_i changes to P_i', $P_i'', \ldots, P_i^{(n)}$. We have

$$\begin{aligned} NP' + N_1 P_1' + \ldots N_n P_n' &= 0, \\ NP'' + N_1 P_1'' + \ldots N_n P_n'' &= 0, \\ &\vdots \\ NP^{(n)} + N_1 P_1^{(n)} + \ldots N_n P_n^{(n)} &= 0. \end{aligned}$$

These relations imply the following condition:

$$\begin{vmatrix} P & P_1 & \cdots & P_n \\ P' & P_1' & \cdots & P_n' \\ P'' & P_1'' & \cdots & P_n'' \\ \vdots & \vdots & \ddots & \vdots \\ P^{(n)} & P_1^{(n)} & \cdots & P_n^{(n)} \end{vmatrix} = 0.$$

By showing that this determinant is different from 0, we show the impossibility of the assumed relation

$$N + e^a N_1 + e^b N_2 + \ldots + e^h N_n = 0.$$

To this end, observe that we can substitute the terms in one row by linear combinations of the other rows, and I will indicate this by considering, for example, the first row. It consists of replacing respectively $P, P_1, \ldots, P_{n-1}$ by $P - e^{-a} P_1$, $e^{-a} P_1 - e^{-b} P_2, \ldots, e^{-g} P_{n-1} - e^{-h} P_n$, $e^{-h} P_n$. It is then easy to see that, upon multiplication of all quantities by $1 \cdot 2 \cdot 3 \cdots \mu$, they yield precisely the integrals

$$\int_0^a e^{-z} f^{\mu}(z) dz, \ \int_a^b e^{-z} f^{\mu}(z) dz, \ \ldots \ \int_g^h e^{-z} f^{\mu}(z) dz, \ \int_h^{\infty} e^{-z} f^{\mu}(z) dz.$$

Now the other rows follow by changing μ to $\mu + 1$, $\mu + 2$, ..., $\mu + n$, and the determinant transforms into the following:

$$\Delta = \begin{vmatrix} \int_0^a e^{-z} f^{\mu}(z)dz & \int_a^b e^{-z} f^{\mu}(z)dz & \cdots & \int_h^\infty e^{-z} f^{\mu}(z)dz \\ \int_0^a e^{-z} f^{\mu+1}(z)dz & \int_a^b e^{-z} f^{\mu+1}(z)dz & \cdots & \int_h^\infty e^{-z} f^{\mu+1}(z)dz \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^a e^{-z} f^{\mu+n}(z)dz & \int_a^b e^{-z} f^{\mu+n}(z)dz & \cdots & \int_h^\infty e^{-z} f^{\mu+n}(z)dz \end{vmatrix}.$$

V. We shall assume, as we have seen before, that μ is a large number. This allows us to determine, by using the beautiful method of Laplace (*De l'integration par approximation des différentielles qui renferment des facteurs élevés à grandes puissance*, in *Théorie analytique des Probabilités*, p. 88), the asymptotic expression of the integrals

$$\int_0^a e^{-z} f^{\mu}(z)dz, \int_a^b e^{-z} f^{\mu}(z)dz, \ldots \int_h^\infty e^{-z} f^{\mu}(z)dz.$$

so that we may find an approximate value for Δ whose ratio with the exact value becomes one for infinite μ. Asuming to this end that the integer numbers $a, b, \ldots, h$ are all positive and ordered by increasing value, such that in each integral the function $e^{-z} f^{\mu}(z)$, which becomes zero in the limits, assumes a single maximum in the interval, I first consider the equation

$$\frac{f'(z)}{f(z)} = \frac{1}{\mu}$$

depending on all these maxima. Now we know that its roots are real and and comprise firstly z_1 between 0 and a, secondly z_2 between a and b, and so on, with the largest z_{n+1} being greater than h. Considered as functions of μ, it is easy to see that they increase for growing μ, and denoting by $p, q, \ldots, s$ the roots of the derived equation $f'(z) = 0$, order by magnitude, we have, if we neglect $\frac{1}{\mu^2}$,

$$z_1 = p + \frac{1}{\mu}\frac{f(p)}{f''(p)}, \quad z_2 = q + \frac{1}{\mu}\frac{f(q)}{f''(q)}, \quad \ldots, \quad z_n = s + \frac{1}{\mu}\frac{f(s)}{f''(s)},$$

and at last $z_{n+1} = (n+1)\mu + \frac{a+b+\ldots+h}{n+1}$, a greater approximation not being necesary. With this, if we write for instance

$$\varphi(z) = \frac{f(z)}{\sqrt{f'^2(z) - f(z)f''(z)}},$$

the desired values will be

$$\sqrt{\frac{2\pi}{\mu}} e^{-z_1} f^{\mu}(z_1)\varphi(z_1), \sqrt{\frac{2\pi}{\mu}} e^{-z_2} f^{\mu}(z_2)\varphi(z_2), \ldots, \sqrt{\frac{2\pi}{\mu}} e^{-z_{n+1}} f^{\mu}(z_{n+1})\varphi(z_{n+1}).$$

But these expressions can be simplified, as we will see.

Considering the first one, observe that we have

$$z_1 = p + \frac{1}{\mu}\frac{f(p)}{f''(p)},$$

where p satisfies the condition $f'(p) = 0$. We conclude that $f(x_1) = f(p)$, only neglecting $\frac{1}{\mu^2}$. Consequently, if we set

$$f(z_1) = f(p)\left(1 + \frac{\alpha}{\mu^2} + \frac{\alpha'}{\mu^3} + \dots\right),$$

then in an analogous way

$$\varphi(z_1) = \varphi(p)\left(1 + \frac{\beta}{\mu} + \frac{\beta'}{\mu} + \dots\right),$$

we have

$$f^{\mu}(z_1) = f^{\mu}(p)\left(1 + \frac{\alpha}{\mu} + \dots\right),$$

and from this we easily deduce

$$f^{\mu}(z_1)\varphi(z_1) = f^{\mu}(p)\varphi(p)\left(1 + \frac{\gamma}{\beta} + \frac{\gamma'}{\mu^2} + \dots\right).$$

In this way, by neglecting only the infinitely small quantities in relation to the preserved terms, we can write

$$\int_0^a \mathrm{e}^{-z} f^{\mu}(z)\mathrm{d}z = \sqrt{\frac{2\pi}{\mu}}\mathrm{e}^{-p} f^{\mu}(p)\varphi(p),$$

and in the same way we have

$$\int_a^b \mathrm{e}^{-z} f^{\mu}(z)\mathrm{d}z = \sqrt{\frac{2\pi}{\mu}}\mathrm{e}^{-q} f^{\mu}(q)\varphi(q),$$

$$\vdots$$

$$\int_g^h \mathrm{e}^{-z} f^{\mu}(z)\mathrm{d}z = \sqrt{\frac{2\pi}{\mu}}\mathrm{e}^{-s} f^{\mu}(s)\varphi(s).$$

But the last integral $\int_h^{\infty} \mathrm{e}^{-z} f^{\mu}(z)\mathrm{d}z$ is of a different analytical form, for the reason that the value $z_{n+1} = (n+1)\mu$ becomes infinite with μ. To obtain it, expand the expression

$$\log(\mathrm{e}^{-z} f^{\mu}(z)\varphi(z))$$

in decreasing powers of the variable, and by neglecting the terms in $\frac{1}{z}, \frac{1}{z^2}, \dots$ we can write

$$\log(f(z)) = (n+1)\log(z), \quad \log(\varphi(z)) = \log\left(\frac{z^{n+1}}{\sqrt{(n+1)z^{2n}+\dots}}\right) = \log\left(\frac{z}{\sqrt{n+1}}\right),$$

and hence

$$\log(\mathrm{e}^{-z} f^{\mu}(z)\varphi(z)) = (n\mu+\mu+1)\log(z) - z - \frac{1}{2}\log(n+1).$$

After substituting the value for z_{n+1}, an easy reduction gives us, when abbreviating

$$\vartheta(\mu) = (n\mu+\mu+1)\log((n+1)\mu) - (n+1)\mu - \frac{1}{2}\log(n+1),$$

the following expression similar to the Euler integral of the first kind,

$$\int_h^{\infty} \mathrm{e}^{-z} f^{\mu}(z)\mathrm{d}z = \sqrt{\frac{2\pi}{\mu}}\mathrm{e}^{\vartheta(\mu)}.$$

Now we will see how the results thus obtained easily lead to the value of the determinant Δ.

VI. I will make a first simplification by omitting the factor $\sqrt{\frac{2\pi}{\mu+i}}$ from the terms in the row i, and then a second one by dividing all the terms in the same column by the first of its entries. The new determinant thus obtained is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ P & Q & \cdots & S & \mathrm{e}^{\vartheta(\mu+1)-\vartheta(\mu)} \\ P^2 & Q^2 & \cdots & S^2 & \mathrm{e}^{\vartheta(\mu+2)-\vartheta(\mu)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P^n & Q^n & \cdots & S^n & \mathrm{e}^{\vartheta(\mu+n)-\vartheta(\mu)} \end{vmatrix},$$

where we write

$$P = f(p),\ Q = f(q),\ \dots,\ S = f(s).$$

Now we see that μ does no longer appear in one column whose terms increase in such a manner that the last one $\mathrm{e}^{\vartheta(\mu+n)-\vartheta(\mu)}$ is infinitely larger than the others. In fact, we have

$$\begin{aligned}\vartheta(\mu+i) &= \vartheta(\mu) + i\vartheta'(\mu) + \frac{i^2}{2}\vartheta''(\mu) + \dots \\ &= \vartheta(\mu) + i\left(\frac{1}{\mu} + (n+1)\log(n+1)\mu\right) + \frac{i^2}{2}\left(-\frac{1}{\mu^2} + \frac{n+1}{\mu}\right) + \dots\end{aligned}$$

and therefore, ignoring $\frac{1}{\mu}, \frac{1}{\mu^2}, \ldots$,

$$\vartheta(\mu + i) - \vartheta(\mu) = i(n+1)\log(n+1)\mu,$$

hence

$$e^{\vartheta(\mu+i)-\vartheta(\mu)} = ((n+1)\mu)^{i(n+1)}.$$

By keeping in the determinant only the term of the highest order in μ, it reduced to this expression:

$$((n+1)\mu)^{n(n+1)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ P & Q & \cdots & S \\ P^2 & Q^2 & \cdots & S^2 \\ \vdots & \vdots & \ddots & \vdots \\ P^{n-1} & Q^{n-1} & \cdots & S^{n-1} \end{vmatrix}.$$

It follows that in general we cannot assume that the determinant Δ vanishes, for the quantities $P = f(p), Q = f(q), \ldots$, integer functions similar to the roots $p, q, \ldots$, have the derived euqation $f'(x) = 0$ whose roots are differnt from each other. This allows us to show the impossibility of any relation of the form

$$N + e^a N_1 + e^b N_2 + \ldots + e^h N_n = 0,$$

and thus we have proved that *the number* e *cannot be a root of an algebraic equation of any degree with integer coefficients*.

But a different path leads to a second, more rigorous proof. In fact, as we will see, by expanding into rational fractions

$$\frac{\Phi_1(x)}{\Phi(x)}, \frac{\Phi_2(x)}{\Phi(x)}, \ldots, \frac{\Phi_n(x)}{\Phi(x)}$$

the method of proof is given by the theory of continous fractions, and completely brings to light the arithmetic character of a non-algebraic irrational number. In this line of ideas, M. Liouville has obtained a remarkable theorem that is the subject of his work titled *Sur des classes très-étendues de quantités dont la valeur n'est ni algébraique, ni même réductible à des irrationelles algébriques* (Comptes rendus, t. XVIII, p. 883 et 910), and I also remind that the famous geometer was the first to prove the proposition that is the subject of these investigations for the case of an equation of degree two and for the biquadratic equation (Journal de Mathématiques, *Note sur l'irrationalité du nombre* e, t. V, p. 192). From the point of view I have taken here, the first proposition has been established.

VII. Let $F(z), F_1(z), \ldots, F_{n+1}(z)$ be the polynomials obtained from the expression

$$z^{\mu}(z-a)^{\mu_1}(z-b)^{\mu_2} \cdots (z-h)^{\mu_n}$$

if we attribute $n+2$ systems of different positive integer values to the exponents $\mu, \mu_1, \ldots, \mu_n$. If $\frac{\Phi_i^k(x)}{\Phi^k(x)}$ represent the fractions converging to the exponentials that correspond to one of these $F_k(z)$, one can always determine the quantities $A, B, C \ldots, L$ by the following equations:

$$\begin{aligned}
A\Phi(x) + B\Phi^1(x) + C\Phi^2(x) + \ldots L\Phi^{n+1}(x) &= 0, \\
A\Phi_1(x) + B\Phi_1^1(x) + C\Phi_1^2(x) + \ldots L\Phi_1^{n+1}(x) &= 0, \\
&\vdots \\
A\Phi_n(x) + B\Phi_n^1(x) + C\Phi_n^2(x) + \ldots L\Phi_n^{n+1}(x) &= 0.
\end{aligned}$$

But instead of deriving from these relations the polynomials $\Phi_i^k(x)$, our objective is to obtain them directly and a priori. For this, I will establish that among the indefinite integrals

$$\int e^{-zx} F(z)\mathrm{d}z, \int e^{-zx} F_1(z)\mathrm{d}z, \ \ldots, \int e^{-zx} F_{n+1}(z)\mathrm{d}z$$

there exists an equation of the form

$$\mathcal{A}\int e^{-zx} F(z)\mathrm{d}z + \mathcal{B}\int e^{-zx} F_1(z)\mathrm{d}z + \ldots + \mathcal{L}\int e^{-zx} F_{n+1}(z)\mathrm{d}z = e^{-zx}\Theta(z),$$

where the coefficients $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{L}$ are independent in z, and $\Theta(z)$ is an integer polynomial divisible by $f(z)$. If we write

$$\mathcal{F}_k(z) = \frac{F_k(z)}{x} + \frac{F_k'(z)}{x^2} + \frac{F_k''(z)}{x^3} + \ldots$$

we have

$$\begin{aligned}
&\mathcal{A}\int e^{-zx} F(z)\mathrm{d}z + \mathcal{B}\int e^{-zx} F_1(z)\mathrm{d}z + \ldots + \mathcal{L}\int e^{-zx} F_{n+1}(z)\mathrm{d}z \\
&= -e^{-zx}(\mathcal{A}\mathcal{F}(z) + \mathcal{B}\mathcal{F}_1(z) + \ldots + \mathcal{L}\mathcal{F}_{n+1}(z)),
\end{aligned}$$

and it is clear that the ratios $\frac{\mathcal{B}}{\mathcal{A}}, \frac{\mathcal{C}}{\mathcal{A}}, \ldots, \frac{\mathcal{L}}{\mathcal{A}}$ can be uniquely determined by assuming the condition that the polynomial

$$\Theta(z) = -(\mathcal{A}\mathcal{F}(z) + \mathcal{B}\mathcal{F}_1(z) + \ldots + \mathcal{L}\mathcal{F}_{n+1}(z))$$

contains $f(z) = z(z-a)(z-b)\cdots(z-h)$ as a factor. By taking the integrals with the limits $z = 0$ and $z = a$, we conclude from this, for example,

$$\mathcal{A}\int_0^a e^{-zx} F(z)\mathrm{d}z + \mathcal{B}\int_0^a e^{-zx} F_1(z)\mathrm{d}z + \ldots + \mathcal{L}\int_0^a e^{-zx} F_{n+1}(z)\mathrm{d}z = 0.$$

Now the relations

$$\int_0^a \mathrm{e}^{-zx} F(z)\mathrm{d}z = \frac{\mathrm{e}^{ax}\Phi(x) - \Phi_1(x)}{\mathrm{e}^{ax}x^{M+1}},$$
$$\int_0^a \mathrm{e}^{-zx} F_1(z)\mathrm{d}z = \frac{\mathrm{e}^{ax}\Phi^1(x) - \Phi_1^1(x)}{\mathrm{e}^{ax}x^{M_1+1}},$$
$$\vdots$$

yield, by separately setting to zero the algebraic term and the coefficients of the exponential e^{ax}, the following equalities

$$A\Phi(x) + B\Phi^1(x) + \ldots + L\Phi^{n+1}(x) = 0,$$
$$A\Phi_1(x) + B\Phi_1^1(x) + \ldots + L\Phi_1^{n+1}(x) = 0.$$

where we abbreviated

$$A = \frac{\mathcal{A}}{x^{M+1}},\ B = \frac{\mathcal{B}}{x^{M_1+1}},\ \ldots,\ L = \frac{\mathcal{L}}{x^{M_{n+1}+1}}.$$

Now we have in the same way, by taking the limits superior of the integrals $z = b, c, \ldots, h$,

$$A\Phi_2(x) + B\Phi_2^1(x) + \ldots + L\Phi_2^{n+1}(x) = 0,$$
$$\vdots$$
$$A\Phi_n(x) + B\Phi_n^1(x) + \ldots + L\Phi_n^{n+1}(x) = 0,$$

and it is easy to see that the coefficients $A, B, \ldots, L$ can be assumed to be integer polynomials in x. The integral $\int_0^1 \mathrm{e}^{-zx} z^m (z-1)^m \mathrm{d}z$ that features in the previously considered relation on p. 4,

$$\mathrm{e}^x \Pi(x) - \Pi_1(x) = \frac{x^{2m+1}\mathrm{e}^x}{1 \cdot 2 \cdot 3 \cdots m} \int_0^1 \mathrm{e}^{-zx} z^m (z-1)^m \mathrm{d}z$$

serves us as a first example.

VIII. In this easy case, where we simply have

$$f(z) = z(z-1),$$

I begin, under the assumption

$$\Theta(z) = xf^{m+1}(z) + (m+1)f^m(z)f'(z),$$

with the following identity:

$$\frac{\mathrm{d}(\mathrm{e}^{-zx}\Theta(z))}{\mathrm{d}z} = \mathrm{e}^{-zx}(\Theta'(z) - x\Theta(z))$$
$$= \mathrm{e}^{-zx}(-x^2 f^{m+1}(z) + (m+1) f^m(z) f''(z) + m(m+1) f^{m-1}(z) f'^2(z)),$$

and I observe that

$$f'^2(z) = 4z^2 - 4z + 1 = 4f(z) + 1, \quad f''(z) = 2,$$

which allows us to write

$$\frac{\mathrm{d}(\mathrm{e}^{-zx}\Theta(z))}{\mathrm{d}z} = \mathrm{e}^{-zx}(-x^2 f^{m+1}(z) + (2m+1)(2m+2) f^m(z) f''(z) + m(m+1) f^{m-1}(z)).$$

We thus have upon integration

$$\mathrm{e}^{-zx}\Theta(z) = -x^2 \int \mathrm{e}^{-zx} f^{m+1}(z)\mathrm{d}z + (2m+1)(2m+2) \int \mathrm{e}^{-zx} f''(z)\mathrm{d}z$$
$$+ m(m+1) \int \mathrm{e}^{-zx} f^{m-1}(z)\mathrm{d}z,$$

and then, if we take the limits $z = 0$ and $z = 1$,

$$x^2 \int_0^1 \mathrm{e}^{-zx} f^{m+1}(z)\mathrm{d}z = (2m+1)(2m+2) \int_0^1 \mathrm{e}^{-zx} f''(z)\mathrm{d}z + m(m+1) \int_0^1 \mathrm{e}^{-zx} f^{m-1}(z)\mathrm{d}z.$$

Now let

$$\varepsilon_m = \frac{x^{2m+1}\mathrm{e}^x}{1 \cdot 2 \cdot 3 \cdots m} \int_0^1 \mathrm{e}^{-zx} z^m (z-1)^m \mathrm{d}z,$$

and this relation becomes

$$\varepsilon_{m+1} = (4m+2)\varepsilon_m + x^2 \varepsilon_{m-1}.$$

This is the result we wish to obtain. By successively taking $m = 1, 2, 3, \ldots$, the resulting equations

$$\varepsilon_2 = 6\varepsilon_1 + x^2\varepsilon_0,$$
$$\varepsilon_3 = 10\varepsilon_2 + x^2\varepsilon_1,$$
$$\varepsilon_4 = 14\varepsilon_3 + x^2\varepsilon_2,$$
$$\vdots$$

easily yield the continued fraction

$$\frac{\varepsilon_1}{\varepsilon_0} = -\cfrac{x^2}{6 + \cfrac{x^2}{10 + \cfrac{x^2}{14 + \ddots}}},$$

and it is sufficient to use the values

$$\varepsilon_0 = x\mathrm{e}^x \int_0^1 \mathrm{e}^{-zx} \mathrm{d}z = \mathrm{e}^x - 1,$$
$$\varepsilon_1 = x^3\mathrm{e}^x \int_0^1 \mathrm{e}^{-zx} z(z-1) \mathrm{d}z = \mathrm{e}^x(2-x) - 2 - x,$$

from which we conclude

$$\frac{\varepsilon_1}{\varepsilon_0} = 2 - \frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1} x,$$

to recover, up to the change from x to $\frac{x}{2}$, the following result of d'Alembert[1)]

$$\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1} = \cfrac{x}{2 + \cfrac{x^2}{6 + \cfrac{x^2}{10 + \cfrac{x^2}{14 + \ddots}}}}.$$

We now consider the general case, and regarding the definite integrals

$$\int_0^a \mathrm{e}^{-z} f^m(z) \mathrm{d}z, \int_0^b \mathrm{e}^{-z} f^m(z) \mathrm{d}z, \ \ldots, \int_0^h \mathrm{e}^{-z} f^m(z) \mathrm{d}z,$$

I propose to obtain an algorithm that allows us to compute them step by step, for all integer numbers m. To make the computations more symmetric, I introduce the following modifications to our previous notations. I write

$$f(z) = (z - z_0)(z - z_1) \cdots (z - z_n)$$

instead of

$$f(z) = z(z-a)(z-b) \cdots (z-h),$$

in the way to consider the most general polynomial of degree $n+1$. Then let Z denote any of the quantities $z_1, z_2, \ldots, z_n$, and I study the integral

$$\int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z) \mathrm{d}z,$$

[1)] Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques, *Mémoires de l'Académie des Sciences de Berlin*, année 1761, p. 265. See also the Note IV of the *Éléments de Géométrie* by Legendre, p. 288.

which evidently includes all those that we have seen before by setting $z_0 = 0$. With this, the path to the method that I want to explain is open.

IX. By integrating the two sides of the identity

$$\frac{\mathrm{d}(\mathrm{e}^{-z} f^m(z))}{\mathrm{d}z} = \mathrm{e}^{-z}(m f^{m-1}(z) f'(z) - f^m(z)),$$

we obtain

$$\mathrm{e}^{-z} f^m(z) = m \int \mathrm{e}^{-z} f^{m-1} f'(z)\mathrm{d}z - \int \mathrm{e}^{-z} f^m(z)\mathrm{d}z,$$

and as a consequence

$$\int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z)\mathrm{d}z = m \int_{z_0}^{Z} \mathrm{e}^{-z} f^{m-1}(z) f'(z)\mathrm{d}z,$$

or also

$$\int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z)\mathrm{d}z = m \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0}\mathrm{d}z + m \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1}\mathrm{d}z + \ldots + m \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n}\mathrm{d}z,$$

by the formula

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_0} + \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_n}.$$

Now there are the new integrals

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0}\mathrm{d}z,\ \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1}\mathrm{d}z,\ \ldots,\ \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n}\mathrm{d}z$$

that give us a sytem of recurrence relations of the form

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0}\mathrm{d}z = (00) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0}\mathrm{d}z + (01) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1}\mathrm{d}z + \ldots + (0n) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n}\mathrm{d}z,$$

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1}\mathrm{d}z = (10) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0}\mathrm{d}z + (11) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1}\mathrm{d}z + \ldots + (1n) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n}\mathrm{d}z,$$

$$\vdots$$

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n}\mathrm{d}z = (n0) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0}\mathrm{d}z + (n1) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1}\mathrm{d}z + \ldots + (nn) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n}\mathrm{d}z,$$

where the coefficients (ij), and thus their determinant, are obtained by a simple procedure, as we will see.

Now, by operating on these $n+1$ elements into which the integral $\int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z)\mathrm{d}z$ decomposes, we will manage to determine its value, rather than by looking for a linear expression of $\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z-z_0}\mathrm{d}z$ via

$$\int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z)\mathrm{d}z,\ \int_{z_0}^{Z} \mathrm{e}^{-z} f^{m+1}(z)\mathrm{d}z,\ \ldots,\ \int_{z_0}^{Z} \mathrm{e}^{-z} f^{m+n}(z)\mathrm{d}z.$$

But, more generally, for arbitrary integer exponents in

$$F(z) = (z-z_0)^{\mu_0}(z-z_1)^{\mu_1}\cdots(z-z_n)^{\mu_n},$$

by integrating the two sides of the identity

$$\frac{\mathrm{d}(\mathrm{e}^{-z}F(z))}{\mathrm{d}z} = \mathrm{e}^{-z}(F'(z)-F(z)),$$

we obtain

$$\mathrm{e}^{-z}F(z) = \int \mathrm{e}^{-z}F'(z)\mathrm{d}z - \int \mathrm{e}^{-z}F(z)\mathrm{d}z,$$

and from that

$$\int_{z_0}^{Z} \mathrm{e}^{-z}F(z)\mathrm{d}z = \int_{z_0}^{Z} \mathrm{e}^{-z}F'(z)\mathrm{d}z.$$

Now the formula

$$\frac{F'(z)}{F(z)} = \frac{\mu_0}{z-z_0} + \frac{\mu_1}{z-z_1} + \ldots + \frac{\mu_n}{z-z_n}$$

gives the following decomposition:

$$\int_{z_0}^{Z} \mathrm{e}^{-z}F(z)\mathrm{d}z = \mu_0\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)}{z-z_0}\mathrm{d}z + \mu_1\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)}{z-z_1}\mathrm{d}z + \ldots + \mu_n\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)}{z-z_n}\mathrm{d}z,$$

which in a similar way leads to an effective computation of the terms

$$\int_{z_0}^{Z} \mathrm{e}^{-z}F(z)\mathrm{d}z,\ \int_{z_0}^{Z} \mathrm{e}^{-z}F(z)f(z)\mathrm{d}z,\ \ldots,\ \int_{z_0}^{Z} \mathrm{e}^{-z}F(z)f^k(z)\mathrm{d}z,$$

and the elements of the decomposition of any of these can be expressed by a linear functon of quantities similar to those that appeared in the preceding term, and this way we will prove it.

X. To this end, I establish that we can always determine two integer polynomials of degree n, say $\Theta(z)$ and $\Theta_1(z)$, such that we have the following relation

$$\int \frac{\mathrm{e}^{-z}F(z)f(z)}{z-\zeta}\mathrm{d}z = \int \frac{\mathrm{e}^{-z}F(z)\Theta_1(z)}{f(z)}\mathrm{d}z - \mathrm{e}^{-z}F(z)\Theta(z).$$

where ζ denotes one of the roots $z_0, z_1, \ldots, z_n$. In fact, after differentiating both sides, we multiply by the factor $\frac{f(z)}{F(z)}$ and obtain

$$\frac{f(z)}{z-\zeta} f(z) = \Theta_1(z) + \left(1 - \frac{F'(z)}{F(z)}\right) f(z)\Theta(z) - f(z)\Theta'(z).$$

Now, as $f(z)$ is divisble by $z - \zeta$, the left hand side in this equality is an integer polynomial of degree $2n+1$, and the right hand side is of the same degree, by the assumptions on $\Theta(z)$, $\Theta_1(z)$. As each of these polynomials contains $n+1$ undetermined coefficients, we certainly have the necessary number of $2n+2$ necessary arbitrary constants in order to make the identification. With this established, observe that when we assume $z = z_i$, then the rational fraction $\frac{F'(z)f(z)}{F(z)}$ takes the value $\mu f'(z_i)$. Thus we obtain the conditions

$$\begin{aligned} \Theta_1(z_0) &= \mu_0 f'(z_0)\Theta(z_0), \\ \Theta_1(z_1) &= \mu_1 f'(z_1)\Theta(z_1), \\ &\vdots \\ \Theta_1(z_n) &= \mu_n f'(z_n)\Theta(z_n), \end{aligned}$$

and this allows, due to the interpolation formula, to immediately calculate $\Theta_1(z)$ if $\Theta(z)$ was known. This way we obtain the following expression:

$$\frac{\Theta_1(z)}{f(z)} = \frac{\mu_0\Theta(z_0)}{z-z_0} + \frac{\mu_1\Theta(z_1)}{z-z_1} + \ldots + \frac{\mu_n\Theta(z_n)}{z-z_n},$$

which will soon be useful. To now obtain $\Theta(z)$, I again consider the proposed relation and divide the two sides by $f(z)$, which yields

$$\frac{f(z)}{z-\zeta} = \frac{\Theta_1(z)}{f(z)} + \left(1 - \frac{F'(z)}{F(z)}\right)\Theta(z) - \Theta'(z),$$

and observe that, as the fraction $\frac{\Theta_1(z)}{f(z)}$ has no integer part, it follows that the desired polynomial has be such that integer part of the expression

$$\left(1 - \frac{F'(z)}{F(z)}\right)\Theta(z) - \Theta'(z)$$

equals that of $\frac{f(z)}{z-\zeta}$. To this effect, first let

$$f(z) = z^{n+1} + p_1 z^n + p_2 z^{n-1} + \ldots + p_{n+1},$$

which gives

$$\frac{f(z)}{z-\zeta} = \begin{array}{r|r|r} z^n + \zeta & z^{n-1} + \zeta^2 & z^{n-2} + \ldots + \zeta^n \\ +p_1 & +p_1\zeta & +p_1\zeta^{n-1} \\ & +p_2 & +p_2\zeta^{n-2} \\ & & \vdots \\ & & +p_n \end{array}$$

or rather

$$\frac{f(z)}{z-\zeta} = z^n + \zeta_1 z^{n-1} + \zeta_2 z^{n-2} + \ldots + \zeta_n,$$

where we write for short

$$\zeta_i = \zeta^i + p_1\zeta^{i-1} + p_2\zeta^{i-2} + \ldots + p_i.$$

Let also

$$\Theta(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \ldots + \alpha_n,$$

and expand the function $\frac{F'(z)}{F(z)}$ in decreasing powers of the variable, in order to obtain the integer part of the product $\frac{F'(z)}{F(z)}\Theta(z)$. If we write $s_i = \mu_0 z_0^i + \mu_1 z_1^i + \ldots + \mu_n z_n^i$, we find in this way that

$$\frac{F'(z)}{F(z)} = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \ldots,$$

and consequently

$$\frac{F'(z)}{F(z)}\Theta(z) = \begin{array}{r|r|l} \alpha_0 s_0 z^{n-1} + \alpha_1 s_0 & z^{n-2} + \alpha_2 s_0 & z^{n-3} + \ldots \\ +\alpha_0 s_1 & +\alpha_1 s_1 & \\ & +\alpha_0 s_2 & \end{array} .$$

The equations in $\alpha_0, \alpha_1, \alpha_2, \ldots$ for which we have made the identification are thus

$$\begin{aligned} 1 &= \alpha_0, \\ \zeta_1 &= \alpha_1 - \alpha_0(s_0 + n), \\ \zeta_2 &= \alpha_2 - \alpha_1(s_0 + n - 1) - \alpha_0 s_1, \\ \zeta_3 &= \alpha_3 - \alpha_2(s_0 + n - 2) - \alpha_1 s_1 - \alpha_0 s_2, \\ &\vdots \end{aligned}$$

They yield

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_1 &= \zeta_1 + s_0 + n, \\ \alpha_2 &= \zeta_2 + (s_0 + n - 1)\zeta_1 + (s_0 + n)(s_0 + n - 1) + s_1, \\ &\vdots \end{aligned}$$

and show that $\alpha_0, \alpha_1, \alpha_2, \dots$ are polynomials in ζ with integer functions as coefficients whose integer coefficients are $s_0, s_1, s_2, \dots$ and therefore have the roots $z_0, z_1, \dots, z_n$. Moreover, we see that α_i is a polynomial of degree i in which the coefficient of ζ_i equals 1. So for simplicity we write

$$\alpha_i = \vartheta_i(\zeta),$$

and from now on write $\Theta(z, \zeta)$ instead of $\Theta(z)$, in order to recognize ζ, so we have

$$\Theta(z, \zeta) = z^n + \vartheta_1(\zeta)z^{n-2} + \vartheta_2(z)z^{n-3} + \dots + \vartheta_n(\zeta).$$

From this results the following formula for the polynomial $\Theta_1(z)$,

$$\frac{\Theta_1(z)}{f(z)} = \frac{\mu_0\Theta(z_0, \zeta)}{z - z_0} + \frac{\mu_1\Theta(z)1, \zeta)}{z - z_1} + \dots + \frac{\mu_n\Theta(z_n, \zeta)}{z - z_n},$$

and from this we can immediately deduce the result we proposed to obtain. In fact, is is sufficient to take the integrals in the limits z_0 and Z in the relation

$$\int \frac{\mathrm{e}^{-z}F(z)f(z)}{z - \zeta}\mathrm{d}z = \int \frac{\mathrm{e}^{-z}F(z)\Theta_1(z)}{f(z)}\mathrm{d}z - \mathrm{e}^{-z}F(z)\Theta(z),$$

which gives

$$\begin{aligned}\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)f(z)}{z - \zeta}\mathrm{d}z &= \int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)\Theta_1(z)}{f(z)}\mathrm{d}z \\ &= \mu_0\Theta(z_0, \zeta)\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)}{z - z_0}\mathrm{d}z \\ &\quad + \mu_1\Theta(z_1, \zeta)\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)}{z - z_1}\mathrm{d}z \\ &\quad \vdots \\ &\quad + \mu_n\Theta(z_n, \zeta)\int_{z_0}^{Z} \frac{\mathrm{e}^{-z}F(z)}{z - z_n}\mathrm{d}z.\end{aligned}$$

In particular, this holds in the case

$$\mu_0 = \mu_1 = \dots = \mu_n = m,$$

in which we will use this equation. If we then set

$$m\Theta(z_i, z_k) = (ik)$$

and successively take ζ equal to $z_0, z_1, \ldots, z_n$, we deduce the previously announced relations, which result from these:

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^{m+1}(z)}{z - z_i} \mathrm{d}z = (i0) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^{m}(z)}{z - z_0} \mathrm{d}z + (i1) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^{m}(z)}{z - z_1} \mathrm{d}z + \ldots + (in) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^{m}(z)}{z - z_n} \mathrm{d}z,$$

for $i = 0, 1, 2, \ldots, n$. However, for the general case I rely on the following proposition.

XI. Let Δ and δ be the determinants

$$\Delta = \begin{vmatrix} \Theta(z_0, z_0) & \Theta(z_1, z_0) & \cdots & \Theta(z_n, z_0) \\ \Theta(z_0, z_1) & \Theta(z_1, z_1) & \cdots & \Theta(z_n, z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Theta(z_0, z_n) & \Theta(z_1, z_n) & \cdots & \Theta(z_n, z_n) \end{vmatrix}$$

and

$$\delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_n \\ z_0^2 & z_1^2 & \cdots & z_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_0^n & z_1^n & \cdots & z_n^n \end{vmatrix}.$$

I claim

$$\Delta = \delta^2.$$

In fact, the expression of $\Theta(z, \zeta)$ in the form

$$\Theta(z, \zeta) = z^n + \vartheta_1(\zeta) z^{n-1} + \vartheta_2(\zeta) z^{n-2} + \ldots + \vartheta_n(\zeta)$$

shows that Δ is the product of the two determinants

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_n \\ z_0^2 & z_1^2 & \cdots & z_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_0^n & z_1^n & \cdots & z_n^n \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \vartheta_1(z_0) & \vartheta_1(z_1) & \cdots & \vartheta_1(z_n) \\ \vartheta_2(z_0) & \vartheta_2(z_1) & \cdots & \vartheta_2(z_n) \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_n(z_0) & \vartheta_n(z_1) & \cdots & \vartheta_n(z_n) \end{vmatrix}.$$

But as $\vartheta_i(\zeta)$ is a polynomial in ζ of degree i, such that we can write

$$\vartheta_i(\zeta) = \zeta^i + r\zeta^{i-1} + s\zeta^{i-2} + \dots,$$

this second quantity reduces to the first by some well-known theorems, and we obtain

$$\Delta = \delta^2.$$

With this established, let

$$\varepsilon = \frac{1}{1 \cdot 2 \cdots m} \int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z) \mathrm{d}z,$$

$$\varepsilon_m^i = \frac{1}{1 \cdot 2 \cdots (m-1)} \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_i} \mathrm{d}z.$$

The relation established on p. 15,

$$\int_{z_0}^{Z} \mathrm{e}^{-z} f^m(z) \mathrm{d}z = m \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0} \mathrm{d}z + m \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1} \mathrm{d}z + \dots + m \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n} \mathrm{d}z,$$

becomes more simply

$$\varepsilon_m = \varepsilon_m^0 + \varepsilon_m^1 + \dots + \varepsilon_m^n,$$

and this again

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^{m+1}(z)}{z - \zeta} \mathrm{d}z = m\Theta(z_0, \zeta) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_0} \mathrm{d}z + m\Theta(z_1, \zeta) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_1} \mathrm{d}z$$
$$+ \dots + m\Theta(z_n, \zeta) \int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f^m(z)}{z - z_n} \mathrm{d}z,$$

by successively taking $\zeta = z_0, z_1, \dots, z_n$, we can make the substitutions denoted by S_m, to have

$$\begin{aligned}
\varepsilon_{m+1}^0 &= \Theta(z_0, z_0)\varepsilon_m^0 + \Theta(z_1, z_0)\varepsilon_m^1 + \dots + \Theta(z_n, z_0)\varepsilon_m^n, \\
\varepsilon_{m+1}^1 &= \Theta(z_0, z_1)\varepsilon_m^0 + \Theta(z_1, z_1)\varepsilon_m^1 + \dots + \Theta(z_n, z_1)\varepsilon_m^n, \\
&\vdots \\
\varepsilon_{m+1}^n &= \Theta(z_0, z_n)\varepsilon_m^0 + \Theta(z_1, z_n)\varepsilon_m^1 + \dots + \Theta(z_n, z_n)\varepsilon_m^n.
\end{aligned}$$

If we now compose $S_1, S_2, \dots, S_{m-1}$, we derive the expressions of $\varepsilon_m^0, \varepsilon_m^1, \dots, \varepsilon_m^n$ in $\varepsilon_1^0, \varepsilon_1^1, \dots, \varepsilon_1^n$ of this form:

$$\begin{aligned}
\varepsilon_m^0 &= A_0\varepsilon_1^0 + A_1\varepsilon_1^1 + \dots + A_n\varepsilon_1^n, \\
\varepsilon_m^1 &= B_0\varepsilon_1^0 + B_1\varepsilon_1^1 + \dots + B_n\varepsilon_1^n, \\
&\vdots \\
\varepsilon_m^n &= L_0\varepsilon_1^0 + L_1\varepsilon_1^1 + \dots + L_n\varepsilon_1^n,
\end{aligned}$$

and the determinant of this new substitution, being equal to the product of the individual substitutions, is $\delta^{2(m-1)}$. It remains to replace $\varepsilon_1^0, \varepsilon_1^1, \ldots, \varepsilon_1^n$, by their values in order for the expressions of the quantities ε_m^i to have the appropriate form for our purpose. These values are easily obtained, as we will see.

XII. To this end, I apply the general formula

$$\int \mathrm{e}^{-z} F(z)\mathrm{d}z = -\mathrm{e}^{-z} \mathcal{F}(z),$$

assuming

$$\mathcal{F}(z) = \frac{F(z)}{z - \zeta}$$

that is,

$$F(z) \;=\; \begin{array}{r|r|r} z^n + \zeta & z^{n-1} + \zeta^2 & z^{n-2} + \ldots + \zeta^n \\ +p_1 & +p_1\zeta & +p_1\zeta^{n-1} \\ & +p_2 & +p_2\zeta^{n-2} \\ & & \vdots \\ & & +p_n \end{array}$$

It is easy to see then that $\mathcal{F}(z)$ becomes an integer expression in z and ζ, very similar to $\Theta(z, \zeta)$, such that, if we denote it by $\Phi(z, \zeta)$, we have

$$\Phi(z, \zeta) = z^n + \varphi_1(\zeta) z^{n-1} + \varphi_2(\zeta) z^{n-2} + \ldots + \varphi_n(\zeta),$$

where $\varphi_i(\zeta)$ is a polynomial in ζ of degree i, in which the coefficient of ζ^i is 1. Thus we obtain, in particular,

$$\begin{aligned} &\varphi_1(\zeta) = \zeta + p_1 + n, \\ &\varphi_2(\zeta) = \zeta^2 + (p_1 + n - 1)\zeta + p_2 + (n-1)p_1 + n(n-1), \\ &\quad\vdots \end{aligned}$$

and the analogy with $\Theta(z, \zeta)$ shows that the determinant

$$\begin{vmatrix} \Phi(z_0, z_0) & \Phi(z_1, z_0) & \cdots & \Phi(z_n, z_0) \\ \Phi(z_0, z_1) & \Phi(z_1, z_1) & \cdots & \Phi(z_n, z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(z_0, z_n) & \Phi(z_1, z_n) & \cdots & \Phi(z_n, z_n) \end{vmatrix}$$

is again equal to δ^2. From this, we deduce the relation

$$\int_{z_0}^{Z} \frac{\mathrm{e}^{-z} f(z)}{z - \zeta} \mathrm{d}z = \mathrm{e}^{-z_0} \Phi(z_0, \zeta) - \mathrm{e}^{-Z} \Phi(Z, \zeta),$$

and by taking $\zeta = z_i$ the desired value

$$\varepsilon_1^i = \mathrm{e}^{-z_0}\Phi(z_0, z_i) - \mathrm{e}^{-Z}\Phi(Z, z_i).$$

But now the expressions for the quantities ε_m^i follow.

Let

$$\begin{aligned}
\mathcal{A} &= A_0\Phi(Z, z_0) + A_1\Phi(Z, z_1) + \ldots + A_n\Phi(Z, z_n),\\
\mathcal{B} &= B_0\Phi(Z, z_0) + B_1\Phi(Z, z_1) + \ldots + B_n\Phi(Z, z_n),\\
&\vdots\\
\mathcal{L} &= L_0\Phi(Z, z_0) + L_1\Phi(Z, z_1) + \ldots + L_n\Phi(Z, z_n),
\end{aligned}$$

and let $\mathcal{A}_0, \mathcal{B}_0, \ldots, \mathcal{L}_0$ represent the values obtained for $Z = z_0$. Then we have

$$\begin{aligned}
\varepsilon_m^0 &= \mathrm{e}^{-z_0}\mathcal{A}_0 - \mathrm{e}^{-Z}\mathcal{A},\\
\varepsilon_m^1 &= \mathrm{e}^{-z_0}\mathcal{B}_0 - \mathrm{e}^{-Z}\mathcal{B},\\
&\vdots\\
\varepsilon_m^n &= \mathrm{e}^{-z_0}\mathcal{L}_0 - \mathrm{e}^{-Z}\mathcal{L}.
\end{aligned}$$

In these formulae, Z denotes any of the quantitites $z_1, z_2, \ldots, z_n$. Now, if we consider this result for $Z = z_k$, we let for this case $\mathcal{A}_k, \mathcal{B}_k, \ldots, \mathcal{L}_k$ and $\eta_k^0, \eta_k^1, \ldots, \eta_k^n$ denote the values of $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{L}$ and $\varepsilon_m^0, \varepsilon_m^1, \ldots, \varepsilon_m^n$, respectively. We obtain the equations

$$\begin{aligned}
\eta_k^0 &= \mathrm{e}^{-z_0}\mathcal{A}_0 - \mathrm{e}^{-z_k}\mathcal{A}_k,\\
\eta_k^1 &= \mathrm{e}^{-z_0}\mathcal{B}_0 - \mathrm{e}^{-z_k}\mathcal{B}_k,\\
&\vdots\\
\eta_k^n &= \mathrm{e}^{-z_0}\mathcal{L}_0 - \mathrm{e}^{-z_k}\mathcal{L}_k,
\end{aligned}$$

which lead us to the anounced second proof of the impossibility of a relation of the form

$$\mathrm{e}^{z_0}N_0 + \mathrm{e}^{z_1}N_1 + \ldots + \mathrm{e}^{z_n}N_n = 0,$$

where the exponents $z_0, z_1, \ldots, z_n$ are assume to be integers, just like the coefficients $N_0, N_1, \ldots, N_n$.

XIII. First, I say that ε_m^i can become smaller than any given quantity for sufficiently large m. In fact, with the exponential e^{-z} being always positive, we have

$$\int_{z_0}^{Z} \mathrm{e}^{-z}F(z)\mathrm{d}z = F(\xi)\int_{z_0}^{Z} \mathrm{e}^{-z}\mathrm{d}z = F(\xi)(\mathrm{e}^{-z_0} - \mathrm{e}^{-Z})$$

for any function $F(z)$ and ξ contained in the interval between z_0 and Z. Now, if we take

$$F(z) = \frac{f^m(z)}{z - z_i},$$

this becomes

$$\varepsilon_m^i = \frac{f^{m-1}(\xi)}{1 \cdot 2 \cdots (m-1)} \frac{f(\xi)}{\xi - z_i}(\mathrm{e}^{-z_0} - \mathrm{e}^{-Z}),$$

which proves the claimed property. From this, I derive from the equations

$$\begin{aligned}
\eta_1^0 &= \mathrm{e}^{-z_0}\mathcal{A}_0 - \mathrm{e}^{-z_1}\mathcal{A}_1, \\
\eta_2^0 &= \mathrm{e}^{-z_0}\mathcal{A}_0 - \mathrm{e}^{-z_2}\mathcal{A}_2, \\
&\vdots \\
\eta_n^0 &= \mathrm{e}^{-z_0}\mathcal{A}_0 - \mathrm{e}^{-z_n}\mathcal{A}_1,
\end{aligned}$$

the following relations:

$$\begin{aligned}
&\mathrm{e}^{z_1}\eta_1^0 N_1 + \mathrm{e}^{z_2}\eta_2^0 + \ldots + \mathrm{e}^{z_n}\eta_n^0 N_n \\
&= \mathrm{e}^{-z_0}(\mathrm{e}^{z_1}N_1 + \mathrm{e}^{z_2}N_2 + \ldots + \mathrm{e}^{z_n}N_n)\mathcal{A}_0 - (\mathcal{A}_1 N_1 + \mathcal{A}_2 N_2 + \ldots + \mathcal{A}_n N_n).
\end{aligned}$$

If we introduce the condition

$$\mathrm{e}^{z_0}N_0 + \mathrm{e}^{z_1}N_1 + \ldots + \mathrm{e}^{z_n}N_n = 0,$$

then this becomes

$$\mathrm{e}^{z_1}\eta_1^0 N_1 + \mathrm{e}^{z_2}\eta_2^0 + \ldots + \mathrm{e}^{z_n}\eta_n^0 N_n = -(\mathcal{A}_1 N_1 + \mathcal{A}_2 N_2 + \ldots + \mathcal{A}_n N_n).$$

Now, by assuming $z_0, z_1, \ldots, z_n$ to be integers, the same holds for the quantities $\Theta(z_i, z_k)$, $\Phi(z_i, z_k)$, and consequently also for $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$. We thus have an integer numbers

$$\mathcal{A}_0 N_0 + \mathcal{A}_1 N_1 + \ldots + \mathcal{A}_n N_n$$

that decreases indefinitely with $\eta_1^0, \eta_1^1, \ldots, \eta_1^n$ for growing m. It then follows that from a certain value for m on, we have

$$\mathcal{A}_0 N_0 + \mathcal{A}_1 N_1 + \ldots + \mathcal{A}_n N_n = 0,$$

and, as we obtain in a similar way the relations,

$$\begin{aligned}
\mathcal{B}_0 N_0 + \mathcal{B}_1 N_1 + \ldots + \mathcal{B}_n N_n &= 0, \\
&\vdots \\
\mathcal{L}_0 N_0 + \mathcal{L}_1 N_1 + \ldots + \mathcal{L}_n N_n &= 0,
\end{aligned}$$

the relation

$$e^{z_0} N_0 + e^{z_1} N_1 + \ldots + e^{z_n} N_n = 0$$

implies that the determinant

$$\Delta = \begin{vmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \cdots & \mathcal{A}_n \\ \mathcal{B}_0 & \mathcal{B}_1 & \cdots & \mathcal{B}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_0 & \mathcal{L}_1 & \cdots & \mathcal{L}_n \end{vmatrix}$$

necessarily has the value 0. But, by the expressions for the quantities $\mathcal{A}_k, \mathcal{B}_k, \ldots, \mathcal{L}_k$, Δ is the product of the two other determinants

$$\begin{vmatrix} A_0 & A_1 & \cdots & A_n \\ B_0 & B_1 & \cdots & B_n \\ \vdots & \vdots & \ddots & \vdots \\ L_0 & L_1 & \cdots & L_n \end{vmatrix}$$

and

$$\begin{vmatrix} \Phi(z_0, z_0) & \Phi(z_0, z_0) & \cdots & \Phi(z_0, z_0) \\ \Phi(z_0, z_1) & \Phi(z_1, z_1) & \cdots & \Phi(z_n, z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(z_0, z_n) & \Phi(z_1, z_n) & \cdots & \Phi(z_n, z_n) \end{vmatrix},$$

of which the first has the value $\delta^{2(m-1)}$ and the second has the value δ^2. We thus have $\Delta = \delta^{2m}$, and it is thus shown in an entirely rigorous manner that the assumed relation is impossible, and as a consequence the number e is not contained in the irrational algebraic numbers.

XIV. It is not without use to give some examples of the mode of approximation of the quantities we have studied, and I consider first the simplest case with only a single exponential e^x. If we set $f(z) = z(z - x)$, we have

$$\varepsilon_m = \frac{1}{1 \cdot 2 \cdots m} \int_0^x e^{-z} z^m (z - x)^m \mathrm{d}z$$

and

$$\varepsilon_m^0 = \frac{1}{1 \cdot 2 \cdots (m-1)} \int_0^x e^{-z} z^{m-1} (z - x)^m \mathrm{d}z$$

$$\varepsilon_m^1 = \frac{1}{1 \cdot 2 \cdots (m-1)} \int_0^x e^{-z} z^m (z - x)^{m-1} \mathrm{d}z.$$

Now we immediately obtain

$$\Theta(z,\zeta) = z + \zeta + 2m + 1 - x,$$

and from this

$$\Theta(0,0) = 2m + 1 - x, \qquad \Theta(x,0) = 2m + 1,$$
$$\Theta(0,x) = 2m + 1, \qquad \Theta(x,x) = 2m + 1 + x,$$

and as a consequence the relations

$$\varepsilon^0_{m+1} = (2m + 1 - x)\varepsilon^0_m + (2m + 1)\varepsilon^1_m,$$
$$\varepsilon^1_{m+1} = (2m + 1)\varepsilon^0_m + (2m + 1_x)\varepsilon^1_m.$$

I observe now that this becomes, by subtracting one from the other,

$$\varepsilon^1_{m+1} - \varepsilon^0_{m+1} = x(\varepsilon^0_m + \varepsilon^1_m),$$

such that with

$$\varepsilon_m = \varepsilon^0_m + \varepsilon^1_m$$

we conclude

$$\varepsilon^1_{m+1} - \varepsilon^0_{m+1} = x\varepsilon_m.$$

By combining this equation with the following,

$$\varepsilon^1_{m+1} + \varepsilon^0_{m+1} = \varepsilon_{m+1},$$

we derive the values

$$\varepsilon^1_{m+1} = \frac{\varepsilon_{m+1} + x\varepsilon_m}{2}, \quad \varepsilon^0_{m+1} = \frac{\varepsilon_{m+1} - x\varepsilon_m}{2},$$

and, if we replave m by $m-1$, then an easy substitution, for example in the relation

$$\varepsilon^0_{m+1} = (2m + 1 - x)\varepsilon^0_m + (2m + 1)\varepsilon^1_m,$$

yields the result previously obtained on p. 13,

$$\varepsilon_{m+1}(4m + 2)\varepsilon_m + x^2\varepsilon_{m-1}.$$

Next, let $n = 2$, $z_0 = 0$, $z_1 = 1$, $z_2 = 2$, so that $f(z) = z(z-1)(z-2) = z^3 - 3z^2 + 2z$. We find

$$\Theta(z,\zeta) = z^2 + (\zeta - 1)z + (\zeta - 1)^2 + 3m(z + \zeta + 1) + 0m^2,$$

and as a consequence,

$$\begin{array}{lll} \Theta(0,0) = 9m^2 + 3m + 1, & \Theta(0,1) = 9m^2 + 6m, & \Theta(0,2) = 9m^2 + 9m + 1, \\ \Theta(1,0) = 9m^2 + 6m + 1, & \Theta(1,1) = 9m^2 + 9m + 1, & \Theta(1,2) = 9m^2 + 12m + 3, \\ \Theta(2,0) = 9m^2 + 9m + 3, & \Theta(2,1) = 9m^2 + 12m + 4, & \Theta(2,2) = 9m^2 + 15m + 7. \end{array}$$

In particular, for $m = 1$, we have

$$\begin{aligned} \varepsilon_2^0 &= 13\varepsilon_1^0 + 16\varepsilon_1^1 + 21\varepsilon_1^2, \\ \varepsilon_2^1 &= 15\varepsilon_1^0 + 19\varepsilon_1^1 + 25\varepsilon_1^2, \\ \varepsilon_2^2 &= 19\varepsilon_1^0 + 24\varepsilon_1^1 + 31\varepsilon_1^2. \end{aligned}$$

Moreover, it is follow easily that

$$\Phi(z,\zeta) = z^2 + (\zeta - 1)z + (\zeta - 1)^2,$$

and this gives

$$\begin{aligned} \varepsilon_1^0 &= 1 - \mathrm{e}^{-Z}(Z^2 - Z + 1), \\ \varepsilon_1^1 &= -\mathrm{e}^{-Z} Z^2, \\ \varepsilon_1^2 &= 1 - \mathrm{e}^{-Z}(Z^2 + Z + 1). \end{aligned}$$

We conclude

$$\begin{aligned} \varepsilon_2^0 &= 34 - \mathrm{e}^{-Z}(50Z^2 + 8Z + 34), \\ \varepsilon_2^1 &= 40 - \mathrm{e}^{-Z}(59Z^2 + 10Z + 40), \\ \varepsilon_2^2 &= 50 - \mathrm{e}^{-Z}(74Z^2 + 12Z + 50). \end{aligned}$$

From this it follows that

$$\begin{aligned} \varepsilon_1 &= \varepsilon_1^0 + \varepsilon_1^1 + \varepsilon_1^2 = 2 - \mathrm{e}^{-Z}(3Z^2 + 2), \\ \varepsilon_2 &= \varepsilon_2^0 + \varepsilon_2^1 + \varepsilon_2^2 = 124 - \mathrm{e}^{-Z}(183Z^2 + 30Z + 124), \end{aligned}$$

and if we successively take $Z = 1$, $Z = 2$, then the expression for ε_1 yields the approximations

$$\mathrm{e} = \frac{5}{2}, \quad \mathrm{e}^2 = \frac{14}{2} = 7,$$

and the expression for ε_2 yields the following,

$$\mathrm{e} = \frac{337}{124}, \quad \mathrm{e}^2 = \frac{916}{124},$$

where the error approaches one over ten-thousand. If we further assume $m = 2$, this gives

$$\begin{aligned}
\varepsilon_3^0 &= 43\varepsilon_2^0 + 49\varepsilon_2^1 + 57\varepsilon_2^2,\\
\varepsilon_3^1 &= 48\varepsilon_2^0 + 55\varepsilon_2^1 + 64\varepsilon_2^2,\\
\varepsilon_3^2 &= 55\varepsilon_2^0 + 63\varepsilon_2^1 + 75\varepsilon_2^2,
\end{aligned}$$

and we obtain

$$\begin{aligned}
\varepsilon_3^0 &= 6272 - \mathrm{e}^{-Z}(9259Z^2 + 1518Z + 6272),\\
\varepsilon_3^1 &= 7032 - \mathrm{e}^{-Z}(10381Z^2 + 1702Z + 7032),\\
\varepsilon_3^2 &= 8140 - \mathrm{e}^{-Z}(12017Z^2 + 1970Z + 8140),
\end{aligned}$$

from which

$$\varepsilon_3 = 21444 - \mathrm{e}^{-Z}(31657Z^2 + 5190Z + 21444),$$

follows, and therefore

$$\mathrm{e} = \frac{58291}{21444}, \quad \mathrm{e}^2 = \frac{158452}{21444},$$

with the error in one over a million.

Original: *Sur la fonction exponentielle*, Gauthier-Villars, Paris, 1874.

Translation by Wolfgang Globke, Version of November 17, 2017.