The inner differential calculus

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I consider it providence that, after a long journey through the realms of algebra and arithmetic, I was once more drawn to differential geometry when the time came to congratulate and pay tribute to the leading German differential geometer on his 75th birthday. This unexpected return demonstrated to me more clearly than any conscious effort might have done how formative the years were which I spent experiencing differential geometry at its source as Prof. Blaschke's assistant. The result of this change of perspective, initiated by contemplation of Dirac's theory, is a calculus that deserved to be placed as *inner differential calculus* next to the established exterior differential calculus, not least because of its relations to complex function theory. I sketched the foundations of this calculus in November 1960 during a Blaschke Festkolloquium, and I believe it serves the matter well if I only now report on this talk in a simplified and extended way.

1 Differentials and differential tensors

The functions $f(x^1, \ldots, x^n)$ that satisfy a certain differentiability condition B on a domain G in n-dimensional space, for example, being infinitely differentiable, form a ring denoted by $\mathscr{r}$. The *exterior differential ring* $\mathcal{R}$ over $\mathscr{r}$ is obtained through extending $\mathscr{r}$ by n symbols $\mathrm{d}x^i$, where $i = 1, \ldots, n$, which are subject only to the relations

$$\mathrm{d}x^i \wedge 1 = 0, \quad \mathrm{d}x^i \wedge a = a \wedge \mathrm{d}x^i \quad \text{for } a \in \mathscr{r},$$
$$\mathrm{d}x^i \wedge \mathrm{d}x^k + \mathrm{d}x^k \wedge \mathrm{d}x^i = 0,$$

where $i, k = 1, \ldots, n$, and the relations following from these. Every element $u \in \mathcal{R}$ can be written in the form

$$u = a + a_i \mathrm{d}x^i + a_{ik} \mathrm{d}x^i \wedge \mathrm{d}x^k + \ldots$$

with $a, a_i, a_{ik}, \ldots \in \mathscr{r}$, where the systems of coefficients $a_{ik}, a_{ikl}, \ldots$ are uniquely determined by u as skew-symmetric tensors, just like a and a_i. Its decomposition

$$u = u_0 + u_1 + \ldots + u_n$$

into homogeneous components u_p of degree p is independent of the coordinate system, which is why

$$\eta u = \sum_{p=0}^{n} (-1)^p u_p, \quad \zeta u = \sum_{p=0}^{n} (-1)^{\binom{p}{2}} u_p$$

define invariant operators η, ζ on $\mathcal{R}$, of which the first one is an automorphism and the second one is an anti-automorphism of $\mathcal{R}$.

Besides the scalar elements of $\mathcal{R}$, which we simply call *differentials*, we also consider *differential tensors*, that is, tensors whose components are elements of $\mathcal{R}$. The action of the operators η, ζ extends componentwise to differential tensors.

Another operator e_l whose meaning only becomes clear when dealing with differential tensors is given by its action

$$e_l u_{i_1 \dots i_p}^{j_1 \cdots j_q} = \sum_{h=0}^{n} \frac{1}{h!} a_{i_1 \dots i_p l k_1 \dots k_h}^{j_1 \cdots j_q} \mathrm{d}x^{k_1} \wedge \cdots \wedge \mathrm{d}x^{k_h},$$

on

$$u_{i_1 \dots i_p}^{j_1 \cdots j_q} = \sum_{h=0}^{n} \frac{1}{h!} a_{i_1 \dots i_p k_1 \dots k_h}^{j_1 \cdots j_q} \mathrm{d}x^{k_1} \wedge \cdots \wedge \mathrm{d}x^{k_h}.$$

While e_l maps differential tensors to differential tensors, the operator $\frac{\partial}{\partial x^l}$, which means partial differentiation of a differential or differential tensor with respect to x^l, depends on the coordinate system. However, if a Riemannian metric is given, covariant differentiation of coefficients with respect to x_l allows us to define an operator d_l which is independent of the coordinate system, commutes with every e_l and gives rise to the definition

$$(\mathrm{d}u)_{i_1 \dots i_p}^{j_1 \cdots j_q} = \mathrm{d}x^l \wedge \mathrm{d}_l u_{i_1 \dots i_p}^{j_1 \cdots j_q}$$

of the *exterior differential* of the differential tensors. Since for a scalar differential u (because of the symmetry in the lower indices of the Chrisoffel symbols) $\mathrm{d}u$ equals $\mathrm{d}x^l \wedge \frac{\partial u}{\partial x^l}$, this is an extension of the exterior differential d defined on the ring of differentials. Its action can be expressed using the differentials repeatedly used by E. Cartan,

$$\omega_i^k = \Gamma_{ij}^k \mathrm{d}x^j$$

as

$$(\mathrm{d}u)_{i_1 \dots i_p}^{j_1 \cdots j_q} = \mathrm{d}x^l \wedge \frac{\partial}{\partial x^i} u_{i_1 \dots i_p}^{j_1 \cdots j_q} + \omega_m^j \wedge u_{i_2 \dots i_p}^{m, j_2 \dots j_q} + \dots - \omega_{i_1}^m \wedge u_{m, i_2 \dots i_p}^{j_1 \cdots j_q}.$$

As an example of a differential tensor let us consider

$$\Omega_i^k = \mathrm{d}(\omega_i^k) - \omega_i^m \wedge \omega_m^k = \frac{1}{2} R_{ijl}^k \mathrm{d}x^j \wedge \mathrm{d}x^l$$

where the parenthesis after d mean that ω_i^k is to be trated like a scalar differential. According to Bianchi, it satisfies

$$(\mathrm{d}\Omega)_i^k = 0.$$

From the simple product rule

$$e_l(u \wedge v) = e_l u \wedge v + \eta u \wedge e_l v \tag{1}$$

we see that the operator e_l has the character of a differentiation as well. This correpsonds precisely to the following equation for exterior differentiation

$$\mathrm{d}(u \wedge v) = \mathrm{d}u \wedge v + \eta u \wedge \mathrm{d}v.$$

In these equations, u, v mean arbitrary differential tensors and $\wedge$ is the exterior tensor product. In the following, we will frequently use this simplification of the notation, unless stated otherwise.

The operators e anti-commute with one another and with η. On the other hand, $e_l \zeta = \eta \zeta e_l$, $e_l \mathrm{d} + \mathrm{d} e_l = \mathrm{d}_l$, $\zeta \mathrm{d} = \mathrm{d} \eta \zeta$, where d and η anti-commute.

We call differential tensors u *constant* that satisfy the equations

$$\mathrm{d}_l u = 0, \quad l = 1, \ldots, n.$$

Products and sums of constant differential tensors are constant. The Bianchi identity tells us in particular that Ω_{ik} is constant. A particularly important role in the theory of Dirac equations is played by the clearly constant volume differential form,

$$z = \sqrt{|g_{ik}|} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$

2 Inner multiplication and differentiation

Whereas the exterior multiplication and differentiation of scalar differentials can be defined without a metric, for the development of the inner differential calclus the existence of a metric fundamental tensor g_{ik} is essential, as the following definition of the *inner product* $u \vee v$ of two differential tensors u, v shows:

$$u \vee v = \sum_{m=0}^{n} (-1)^{\binom{m}{2}} \frac{\eta^m}{m!} e_{i_1} \cdots e_{i_m} u \wedge e^{i_1} \cdots e^{i_m} v \quad (e^i = g^{ik} e_k).$$

We may call this a multiplication by virtue of the associativity law $(u \vee v) \vee w = u \vee (v \vee w)$.

Contrary to the exterior multiplication, for which the simple rule

$$u \wedge v = (-1)^p v \wedge u \quad \text{(for } v \text{ homogeneous of degree } p\text{)}$$

holds, the inner multiplication is non-commutative in an intricate way, as the following analogue of the above equation shows:

$$u \vee v = \sum_{m=0}^{n} (-1)^{\binom{m}{2}} \frac{2^m}{m!} e_{i_1} \cdots e_{i_m} \eta^{p+m} \vee e^{i_1} \cdots e^{i_m} u$$

where u is homogeneous of degree p.

Nevertheless, performing inner multiplications in physical applications is often easy, since in the case of all g_{ik} vanishing for $i \neq k$, all the inner products $dx^i \vee dx^k \vee \cdots \vee dx^l$ with all indices distinct coincide with the corresponding exterior products.

The relation

$$dx^i \vee dx^k = dx^i \vee dx^k + g^{ik}$$

allows to conclude by virtue of its consequence $dx^i \vee dx^k + dx^i \vee dx^k = 2g^{ik}$, that the inner multiplication of scalar differentials is the Clifford multiplication corresponding to the symmetric matrix g_{ik}. This implies that the "exterior" differential ring R is a ring in a second way, in which it is called the *inner differential ring*. For this ring as well η is an automorphism, ζ is an anti-automorphism and the formula corresponding to (1) holds,

$$e_l(u \vee v) = e_l u \vee v + \eta u \vee e_l v.$$

Right-multiplication with the volume differential z is the transfer of a differential u to its dual $*u = i \vee z$ which is so important in the theory of Hodge.

The interpretation of $du = dx^l \wedge d_l u$ of exterior differentiation suggests to define as its analogue the *inner differentiation*

$$\delta u = dx^l \vee d_l u.$$

Indeed, this defines an operation which at least by virtue of its appearance in the Dirac equation and as the square root of the Laplace-Beltrami operator $\Delta = \delta\delta$ merits our attention.

Its connection to the exterior differential is given through the equation

$$\delta u = du + e^l d_l u,$$

which can also be written

$$\delta u = \mathrm{d}u + \mathrm{d}^* u \quad \text{with} \quad \mathrm{d}^* u = *^{-1}\mathrm{d} * u = (-1)^{\binom{n}{2}}\mathrm{d}(u \vee z) \vee z$$

and thus highlights the relation to the operator Δ in the case of scalar differentials via the then valid relation $\mathrm{d}^*\mathrm{d}^* u = 0$. It is important in relation to the theory of Dirac equations that a rather simple product rule

$$\delta(u \vee v) = (\delta u) \vee v + \eta u \vee (\delta v) + 2e^n u \vee \mathrm{d}_h v. \tag{2}$$

holds.

The mirror image of inner differentiation is

$$\zeta \delta \zeta u = \mathrm{d}_l u \vee \mathrm{d}x^l.$$

Both δ and d commute with the operator η, and it also holds that

$$\delta e_l + e_l \delta = \mathrm{d}_l$$

in analogy to a relation between d, e and d_l mentioned above.

3 Scalar products

Out of two differentials u, v we obtain a differential

$$(u, v) = (\zeta u \vee v) \wedge z$$

which differs from the volume differential z by the factor $(\zeta u \vee v)_0$ that comprises the terms of degree 0 in the decomposition $\sum_{m=0}^{n} (\zeta u \vee v)_m$ of $\zeta u \vee v$ into homogeneous differentials $(\zeta u \vee v)_m$ of degree m.

Even though it would be more appropriate to call the integral over the whole space of (u, v) the scalar product of u and v, we will, in regard to non-compact spaces, call the differential (u, v) itself the *scalar product* of u and v. It has the properties

$$(u, v) = (v, u) = (\eta u \eta v) = (\zeta u, \zeta v) = (u \vee v, v \vee z) = (u \vee z, v \vee z) = (z \vee u, z \vee v),$$

and for arbitrary differentials u, v, w we have

$$\begin{aligned} (u \vee w, v) &= (u, v \vee \zeta w), \\ (w \vee u, v) &= (u \zeta w \vee v). \end{aligned}$$

Its computation is simplified by the observation that from the decomposition into homogeneous parts, $u = \sum_{m=0}^{n} u_m$, $v = \sum_{m=0}^{n} v_m$, we obtain

$$(u, v) = \sum_m (u_m, v_m) = \sum_m (\zeta u_m \wedge *v_m),$$

where $*v_m = v_m \vee z$.

Besides the n-fold differential (u, v), the $n - p$-fold differentials

$$(u, v)_p = \frac{1}{p!} e_{i_1} \cdots e_{i_p} (\mathrm{d}x^{i_p} \vee \cdots \vee \mathrm{d}x^{i_1} \vee u, v)$$

deserve our attention. They have similar properties

$$(v, u) = (-1)^{\binom{p}{2}} (u, v)_p, \quad (\eta u, \eta v)_p = (-1)^p (u, v)_p,$$
$$(u \vee w, v)_p = (u, v \vee \zeta w)_p,$$

and

$$(u, v)_1 = e_i (\mathrm{d}x^i \vee u, v) = (\zeta u \vee \mathrm{d}x^i \vee v)_0 e_i z$$

stands out by its appearance in *Green's formula*

$$\mathrm{d}(u, v)_1 = (u, \delta v) + (v, \delta u).$$

Note that in the case of a positive definite metric, (u, u) differs from the volume differential z by a factor that is positive precisely where u does not vanish, that is, where at least one coefficient of u is different from 0.

4 Lie operators and differentials

A Lie operator acting on functions,

$$A = \alpha^i (x^1, \ldots, x^n) \frac{\partial}{\partial x^i}$$

defines a contravariant tensor α^i with the help of which we can form an expression

$$Au = \alpha^i \frac{\partial u}{\partial x^i} + \mathrm{d}(\alpha^i) \wedge e_i u$$

independently of the coordinate system. In the presence of a metric, this expression coincides with

$$Au = \alpha^i \cdot \mathrm{d}_i u + (\mathrm{d}\alpha)^i \wedge e_i u.$$

In the first of these formulas, the exterior differential $\mathrm{d}(\alpha^i)$ of the ith component of the tensor α^i appears, whereas $(\mathrm{d}\alpha)^i$ denotes the ith component of the exterior differential of the tensor α^i. The operator A thus declared on the ring $\mathcal{R}$ of differentials has the properties

$$A(u \wedge v) = Au \wedge v + u \wedge Av, \quad \mathrm{d}Au = A\mathrm{d}u, \tag{3}$$

and if three of these operators A, B, C acting on functions are in the relation $AB - BA = C$, then the same holds for their extensions A, B, C to operators in the ring $\mathcal{R}$.

In the presence of the metric every Lie operator gives rise to a scalar differential $\alpha = \alpha_i \cdot \mathrm{d}x^i = g_{ik}\alpha^k \mathrm{d}x^i$.

The *Killing equations*

$$\mathrm{d}_i\alpha_k + \mathrm{d}_k\alpha_i = 0$$

hold if and only if the metric at A is invariant in the following sense: If we introduce coordinates in which $\alpha^i = 0$ for $i < n$, $\alpha^n = 1$, then the coefficients of the fundamental tensor g_{ik} are indepdendent of x^n. Under these assumptions, the action of the operator A can be described by

$$Au = \alpha^i \cdot \mathrm{d}_i u + \frac{1}{4}(\mathrm{d}\alpha \vee u - u \vee \mathrm{d}\alpha)$$

and the analogous relations to (3)

$$A(u \vee v) = Au \vee v + u \vee Av, \quad \delta A$$

hold.

If the differentials α, β, γ are associated to the three operators A, B, C that leave the metric invariant and are related by $AB - BA = C$, then

$$\mathrm{d}\gamma = \frac{1}{4}(\mathrm{d}\alpha \vee \mathrm{d}\beta - \delta\beta \vee \delta\alpha) + 2\alpha^i \cdot \beta^k \cdot \Omega_{ik}.$$

5 Dirac equations

The inner differential calculus finds its raison d'être in the fact that important systems of partial differential equations take the form

$$\delta u = a \vee u,$$

where a is a given and u the desired differential.

We call every equation of this form a *Dirac equation*, since Dirac's theory of the states of electrons in an electromagnetic field gave rise to the study of such equations in the first place.

From the product rule (2) it follows that a solution u of a Dirac equation gives rise to another solution $u \vee c$ of the same Dirac equation by inner right-multiplication by an arbitrary constant differential.

Every operator that, like $\vee c$, maps solutions of a Dirac equation to solutions of the same Dirac equation is called an *integral of the Dirac equation*. For example, every Dirac equation has the transition $* = \vee z$ to the dual as an integral.

If a one-parameter group determined by the operator A preserves the metric and the differential a, which is equivalent to A satisfying the Killing equations and the condition $Aa = 0$, then A is an integral of the Dirac equation.

If u is a solution of the Dirac equation $\delta u = a \vee u$ and v is a solution of the *adjoint Dirac equation*

$$\delta v = -\zeta a \vee v,$$

then

$$\mathrm{d}(u, v)_1 = 0,$$

which follows from Green's formula.

6 Spherical differentials

In three-dimensional Euclidean space, the constant differentials are precisely those that have constant coefficients when written as inner or exterior polynomials in $\mathrm{d}x^1$, $\mathrm{d}x^2$, $\mathrm{d}x^3$. In particular,

$$w = \mathrm{d}x^1 \vee \mathrm{d}x^2 \vee \mathrm{d}x^3 \quad \text{and} \quad w_i = \mathrm{d}x^i \vee w = w \vee \mathrm{d}x^i$$

are constant. The rotations about the three coordinate axes give rise to three operators X_1, X_2, X_3 whose action on differentials w is given by

$$X_i u = x^k \frac{\partial u}{\partial x^l} - x^l \frac{\partial u}{\partial x^k} + \frac{1}{2} w_i \vee v - \frac{1}{2} u \vee w_i,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Like the well-known Lie operators they satisfy the relation

$$X_k X_l - X_l X_k = -X_i$$

and commute with the operator K given by

$$(K + 1)u = \sum_i X_i u \vee w_i$$

independently of the coordinate system, with which they also satisfy the relation

$$X_1^2 + X_2^2 + X_3^2 = -K - K^2.$$

K also commutes with the operator $\vee w$:

$$K(u \vee w) = Ku \vee w,$$

and in polar coordinates, Ku is best determined via the formula

$$(K+1)u = -\zeta\delta\zeta u \vee r\mathrm{d}r + \sum_{i=1}^{3} x^i \frac{\partial u}{\partial x^i} + \frac{3}{2}(u - \eta u) + g\eta u,$$

in which g denotes the linear operator that simply mutliplies every homogeneous differential by its degree.

The *spherically symmetric* (space) differentials are precisely those differentials u that satisfy the equations

$$X_1 u = X_2 u = X_3 u = 0,$$

or equivalently, those that can be written as inner polynomials in $\mathrm{d}r$, w with coefficients depending only on r. For every Dirac equation $\delta u = a \vee u$ with spherically symmetric a, the operators X_1, X_2, X_3, K, $\vee w$, and of course all other operators $\vee c$ with constant c, are integrals. In particular, this holds for the Dirac equation

$$\delta u = 0,$$

whose solution leads to the important spherical differentials if we ask for those solutions of this equation that are rationally homogeneous in x^1, x^2, x^3.

By means of the Legendre polynomials P_k^m, which may also have negative lower index by setting $P_k^m = P_{-k-1}^m$, we first form the expression

$$Y_k^m = P_k^m(\cos(\theta))\mathrm{e}^{\mathrm{i}m\varphi}$$

in polar coordinates r, θ, φ, and then form the *spherical differential*

$$S_k^m = r^{1-k} \cdot \mathrm{d}(r^k \cdot Y_k^m),$$

which satisfies the Dirac equation

$$\delta S_k^m = \frac{1-k}{r}\mathrm{d}r \vee S_k^m$$

equivalent to $\delta(r^{k-1} \cdot S_k^m) = 0$, and is a K-eigendifferential with eigenvalue k,

$$KS_k^m = kS_k^m.$$

It is also an eigendifferential of the operator X_3:

$$X_3 S_k^m = \mathrm{i} m S_k^m,$$

and for applications it is important that for every spherically symmetric differential R,

$$\delta(R \vee S_k^m) = \left(\delta R + \eta\zeta R \vee \frac{1-k}{r}\mathrm{d}r\right) \vee S_k^m$$

holds.

The meaning of the spherical differentials for the integration of the Dirac equation

$$\delta u = 0$$

becomes clear from the following observation: The solutions of the equation that are twice differentiable on the whole space except the origin can be expanded into a uniformly convergent series

$$u = \sum R_k^m \vee S_k^m$$

on every compact set excluding the origin, where R_k^m is spherically symmetric, namely,

$$ar^{k-1} + a'r^{k-1}w + a''r^{-k-1}\mathrm{d}r + a'''r^{-k-1}\mathrm{d}r \vee w$$

with constants a, a', a'', a'''.

7 Dirac equations in space and time

The Einstein-Minkowski metric

$$(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - \mathrm{d}c^2(\mathrm{d}t)^2$$

determines an inner differential calculus in which $\mathrm{d}x^1$, $\mathrm{d}x^2$, $\mathrm{d}x^3$ satisfy the same relations as in the space considered above, whereas computations involving $\mathrm{d}t$ are subject to the following rules:

$$\mathrm{d}x^i \vee \mathrm{d}t = -\mathrm{d}t \vee \mathrm{d}x^i = \mathrm{d}x^i \wedge \mathrm{d}t, \quad \mathrm{d}t \vee \mathrm{d}t = -c^2.$$

Those differentials that do not contain $\mathrm{d}t$ when written as exterior or inner polynomials in $\mathrm{d}x^1$, $\mathrm{d}x^2$, $\mathrm{d}x^3$, $\mathrm{d}t$ will be called *space differentials*. We speak of a *pure space differential* if the coefficients in such a representation do not depend on t.

The totality of all pure space differentials can be considered as an inner (and also exterior) subring of the ring of all differentials of Einstein-Minkowski space, and here the inner differentiation is the extension of the inner differentiation on Euclidean space, and so both shall be denoted by δ. Keeping the other notations from the previous section as well, we can decompose the volume differential z into

$$z = \mathrm{d}x^1 \vee \mathrm{d}x^2 \vee \mathrm{d}x^3 \vee \mathrm{i}c\mathrm{d}t = w \vee \mathrm{i}c\mathrm{d}t = w \wedge \mathrm{i}c\mathrm{d}t.$$

Its inner square is $z \vee z = 1$.

Just like z, every other inner or exterior polynomial in $\mathrm{d}x^1$, $\mathrm{d}x^2$, $\mathrm{d}x^3$, $\mathrm{d}t$ with constant coefficients is constant, and there are no other constant differentials.

The constant differentials

$$\varepsilon^{\pm} = \frac{1}{2} \mp \frac{\mathrm{i}c}{2}\mathrm{d}t$$

satisfying the relations $\varepsilon^{\pm} \vee \varepsilon^{\pm} = \varepsilon^{\pm}$, $\varepsilon^{\pm} \vee \varepsilon^{\mp} = 0$ and $\varepsilon^{+} + \varepsilon^{-} = 1$, give rise to a decomposition of an arbitrary differential u into two summands $u^{+} \vee \varepsilon^{+}$ and $u^{-} \vee \varepsilon^{-}$ (with space differentials u^{+}, u^{-}) that are uniquely determined by u as the eigendifferentials of the idempotent operators $\vee\varepsilon^{+}$ and $\vee\varepsilon^{-}$.

For every Dirac equation

$$\delta u = a \vee u$$

with $a = \alpha + \beta \vee \mathrm{i}c\mathrm{d}t$ (and α, β space differentials), in which the differential a does not depend on t in the sense that $\frac{\partial a}{\partial t} = 0$, the *energy operator* H given by

$$Hu = -\frac{h}{2\pi \mathrm{i}} \frac{\partial u}{\partial t}$$

is an integral, and as H commutes with the integrals $\vee\varepsilon^{+}$, $\vee\varepsilon^{-}$, the complete solutions of such a Dirac equation can be reduced to the determination of the simultaneous eigendifferentials of H and $\vee\varepsilon^{\pm}$, that is, those the differentials

$$u = p \vee T^{\pm} \quad \text{with } T^{\pm} = \varepsilon^{\pm} \cdot \mathrm{e}^{-\frac{2\pi \mathrm{i}}{h} E t},$$

where the first factor, as a pure space differential, must satisfy

$$\delta p = \alpha \vee p \mp \left(\frac{2\pi}{hc} E + \beta\right) \vee \eta p. \tag{4}$$

8 Spherically symmetric Dirac equations

Among the Dirac equations in space and time with energy integral, those in which α and β are spherically symmetric space differentials distinguish themselves by physical applicability and the possibility of reduction to ordinary differential equations.

In this case we can continue further the above separation of u into a pure space differential p and a pure time differential T by setting

$$p = R \vee S$$

and require that R is spherically symmetric and $S = S_k^m$ is a spherical differential. Indeed, it is sufficient to determine R such that

$$\delta R = \alpha \vee R \pm \left(\frac{2\pi}{hc}E + \beta\right) \vee \eta R + \frac{k-1}{r}\mathrm{d}r \vee \eta\zeta R$$

holds, which by making the ansatz $R = f_0(r) + f_1(r)\mathrm{d}r + f_2(r)\mathrm{d}w + f_3(r)\mathrm{d}r \vee \mathrm{d}w$ leads to the following system of ordinary differential equations:

$$\frac{\mathrm{d}f_1}{\mathrm{d}r} + \frac{1+k}{r}f_1 \mp \frac{2\pi}{hc}Ef_0 = (\alpha_0 \pm \beta_0)f_0 + (\alpha_1 \mp \beta_1)f_1 + (-\alpha_2 \pm \beta_2)f_2 + (-\alpha_3 \mp \beta_3)f_3,$$

$$\frac{\mathrm{d}f_0}{\mathrm{d}r} + \frac{1-k}{r}f_0 \pm \frac{2\pi}{hc}Ef_1 = (\alpha_1 \pm \beta_1)f_0 + (\alpha_0 \mp \beta_0)f_1 + (-\alpha_3 \pm \beta_3)f_2 + (-\alpha_2 \mp \beta_2)f_3,$$

$$\frac{\mathrm{d}f_2}{\mathrm{d}r} + \frac{1+k}{r}f_3 \pm \frac{2\pi}{hc}Ef_2 = (\alpha_2 \pm \beta_2)f_0 + (\alpha_3 \mp \beta_3)f_1 + (\alpha_0 \mp \beta_0)f_2 + (\alpha_1 \pm \beta_1)f_3,$$

$$\frac{\mathrm{d}f_2}{\mathrm{d}r} + \frac{1-k}{r}f_1 \mp \frac{2\pi}{hc}Ef_3 = (\alpha_3 \pm \beta_3)f_0 + (\alpha_2 \mp \beta_2)f_1 + (\alpha_1 \mp \beta_1)f_2 + (\alpha_0 \mp \beta_0)f_3 \tag{5}$$

where we set

$$\alpha = \alpha_0(r) + \alpha_1(r)\mathrm{d}r + \alpha_2(r)w + \alpha_3(r)\mathrm{d}r \vee w,$$
$$\beta = \beta_0(r) + \beta_1(r)\mathrm{d}r + \beta_2(r)w + \beta_3\mathrm{d}r \vee w.$$

9 The Dirac equation of the electron

The relation between the vector potential A_1, A_2, A_3, the electric potential Φ and the electromagnetic field can be described, after introducing the *field differential*

$$\omega = A_1\mathrm{d}x^1 + A_2\mathrm{d}x^2 + A_3\mathrm{d}x^3 - c\Phi\mathrm{d}t,$$

by

$$\mathrm{d}\omega = \Theta,$$

where

$$\Theta = H_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + H_2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + H_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2$$
$$+ cE_1 \mathrm{d}x^1 \wedge \mathrm{d}t + cE_2 \mathrm{d}x^2 \wedge \mathrm{d}t + cE_3 \mathrm{d}x^3 \wedge \mathrm{d}t$$

satisfies the equations

$$\mathrm{d}\Theta = 0, \quad \delta\Theta = 0$$

equivalent to the Maxwell equations. Since only $\mathrm{d}\omega$ has a physical meaning, all physical statements derived from ω must be *gauge invariant*, that is, invariant under the transformation $\omega \mapsto \omega + \mathrm{d}f$. If the arbitrariness in the choice of ω is further constrained by the *Lorentz condition*,

$$\mathrm{d}\omega = \delta\omega,$$

then ω is harmonic in the sense that $\delta\delta\omega = 0$.

Since electrodynamics allows such a concise formulation in the language of differentials, it suggests itself to consider the Dirac theory of the electron as an affair of inner differential calculus.

The *spin* of an electron is considered as an unmistakable sign that the states of an electron cannot be described by a state function, but by a *state differential*. In an electromagnetic field given by the field differential ω only those states are possible whose differential u satisfies the Dirac equation

$$\frac{h}{2\pi \mathrm{i}} \delta u = \frac{1}{c}(\mathrm{i}E_0 \pm e\omega) \vee u,$$

with E_0 the rest energy and e the charge of an electron. Namely, the states of the "negative" electron are given by the condition

$$u \vee \varepsilon^- = u, \quad u \vee \varepsilon^+ = 0,$$

and those of the positron by

$$u \vee \varepsilon^+ = u, \quad u \vee \varepsilon^- = 0.$$

So these are the (automatically simultaneous) eigendifferentials of the integrals $\vee\varepsilon^+$, $\vee\varepsilon^-$ of the Dirac equation.

What is physically relevant is not the state differential u itself, but the following triple differential derived from it,

$$|e| \cdot (u, \eta\overline{u})_1 = \varrho \cdot w - (i_1 \cdot w_1 + i_2 \cdot w_2 + i_3 \cdot w_3) \wedge \mathrm{d}t,$$

which determines the *current density* (i_1, i_2, i_3) and the *charge density* ϱ.

This interpretation is possible by the conservation law

$$\mathrm{d}(u, \eta\overline{v})_1 = 0$$

that holds for any pair of solutions u, v of the above Dirac equation.

As befits every physically relevant differential, the density differential $|e|\cdot(u, \eta\overline{u})_1$ is gauge invariant in the following sense: If we replace ω by $\omega + \mathrm{d}f$ in the Dirac equation, then the new Dirac equation is solved by

$$v = \mathrm{e}^{\frac{2\pi\mathrm{i}}{hc}ef} u$$

if u is a solution of the original Dirac equation. The requirement of gauge invariance of the density differential thus is satisfied if every change $\omega \mapsto \omega + \mathrm{d}f$ of the field differential is accompanied by a change

$$u \mapsto \mathrm{e}^{\frac{2\pi\mathrm{i}}{hc}ef} u$$

of the state differential.

The interpretation of Dirac's theory presented here also holds up when the states of an electron in a spherically symmetric Coulomb field are to be determined, that is, for $\omega = -c\Phi\mathrm{d}t$ with $\Phi = \frac{Z}{r}|e|$. With the method indicated above we obtain state differentials of the form

$$u = R \vee S \vee T,$$

where R is a spherically symmetric space differential, S a spherical differential, and T time differential that contains the energy constant. Since in the present situation $\alpha = \alpha_0 = -\frac{2\pi}{hc}E_0$, $\alpha_k = 0$ for $k \neq 0$, $\beta = \beta_0 = \frac{2\pi}{hc}\frac{Ze^2}{r}$, $\beta_k = 0$ for $k \neq 0$, the differential equations (5) simplfy considerably, namely to the two known radial differential equations, if we consider that due to the absence of w in α and β, R can be assumed as to be independent of w as well (that is, $f_2 = f_3 = 0$) or only contains w as a factor, in which case $f_0 = f_1 = 0$ holds.

It deserves to be noted that in this calculus, $\frac{h}{2\pi\mathrm{i}}X_k$, $k = 1, 2, 3$, is naturally interpreted as the total angular momentum, because the verbatim translation of the usual interpretation would require $\frac{h}{2\pi\mathrm{i}}(X_k + \frac{1}{2}w_k)$ instead.

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