On closed orbits of reductive algebraic groups

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The base field K is algebraically closed and of characteristic zero. We follow the notation of [1].

Our goal is to prove the following theorem:

Theorem *Let G be a reductive algebraic group that operates morphically on a smooth affine algebraic variety X. Assume that at every point of X the tangent space admits a non-degenerate symmetric bilinear form that is invariant under the isotropy subgroup. Then there exists a dense open subset of X consisting of closed orbits of X.*

The assumption of the theorem are necessary, for example:

For the adjoint action of a reductive group (in this case, the conclusion is well-known, see [2]).

For $G = H_2$ and $X = H/H_1$, where $H_1, H_2 \subset H$ are reductive groups (for more details, see the end of paragraph 3).

Corollary 1 *Under the hypotheses of the Theorem, there exists a dense open subset of X on which the isotropy subgroup of G is reductive.*

In fact, every closed orbit in X is affine, and by a result of Matsushima [3], its isotropy subgroup then is reductive.

Corollary 2 *Under the hypotheses of the Theorem, every open orbit is closed.*

1 Étale slices

Let G be a reductive group that operates on an affine variety X. Let $x \in X$. Let G_x denote the isotropy subgroup in G at x and $G(x)$ the orbit of G passing through x. We assume that G_x is reductive (if G is reductive, this amounts to the assumption that $G(x)$ is affine [3]). We denote by $K[X]$ the algebra of regular functions on X.

Lemma 1 *If X is smooth at x, then there exists a morphism $\varphi : X \to \mathrm{T}_x X$ of varieties with the following properties:*

(1) φ *commutes with the action of* G_x,

(2) φ *is étale at* x,

(3) $\varphi(x) = 0$.

PROOF: Let $\mathfrak{m}$ denote the maximal ideal of $K[X]$ that corresponds to the point x. The canonical map $d : \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 = (\mathrm{T}_x X)^*$ commutes with the action of G_x. As G_x operates completely reducibly on $K[X]$ ([5, Chapter 1, §1]), we can find a G_x-submodule W of $\mathfrak{m}$ such that $d : W \to (\mathrm{T}_x X)^*$ is an isomorphism. Prolong $(d|_W)^{-1}$ in a canonical way to a homomorphism from the symmetric algebra of $(\mathrm{T}_x X)^*$ to $K[X]$. One easily verifies that the corresponding morphism $\varphi : X \to \mathrm{T}_x X$ satisfies the requirements of the lemma. ◇

Set $Y = \varphi^{-1}(N)$. This is a closed subvariety of X containing x, smooth at x, invariant by G_x and such that $\mathrm{T}_x Y = N$. The group G_x acts on $G \times Y$ by $s(t, y) = (ts^{-1}, sy)$, and hence also on $K[G \times Y]$. As G_x is reductive, it acts completely reducibly on $K[G \times Y]$, from which we deduce that the algebra of invariants $K[G \times Y]^{G_x}$ is of finite type over K, see [5]. Let $G \times_{G_x} Y$ be the affine variety defined by $K[G \times_{G_x} Y] = K[G \times Y]^{G_x}$. We verfiy that $G \times_{G_x} Y$ is the fibration associated with the G_x-principal fibration G/G_x and fiber of type Y. Denote by e the identity element of G and by $\overline{(e, x)}$ the image of the point $(e, x) \in G \times Y$ in $G \times_{G_x} Y$. Since X is smooth at x, $G \times_{G_x} Y$ is smooth at $\overline{(e, x)}$. The action of G on $G \times Y$ given by $s(t, y) = (st, y)$ descends to an action of G on $G \times_{G_x} Y$. The morphism $G \times X \to X$ that defines the G-action on X, induces a morphism $G \times_{G_x} Y \to X$ which commutes with the G-action. Since $\mathrm{T}_x G(x) + \mathrm{T}_x Y = \mathrm{T}_x X$ and since $G \times_{G_x} Y$ and X have the same dimension, $G \times_{G_x} Y \to X$ is étale at the point $\overline{(e, x)}$. Let $V = V(\varphi, N)$ denote the largest open subset of Y such that $\varphi : V \to N$ and $G \times_{G_x} V \to X$ are étale. Let $U = U(\varphi, N)$ denote the image of $G \times_{G_x} V$ in X; this is an open subset that is stable by G and contains x.

2 Closed orbits

Let G be an algebraic group which acts on an algebraic variety X. We say that *almost all* orbits of G in X are closed if there exists on open dense subset of X consisting of closed orbits (in X). This notion has already been studied in several ways (see [8]).

The function which associates to a point $x \in X$ the dimension of the orbit passing through x is lower-semicontinuous (see [5, p. 7]). We denote by $A = A(X)$ the set of points in X where it is not locally constant. We further denote by

$B = B(X)$ the set of points in X through which passes an orbit whose closure intersects A. If G is reductive and X affine, then B is closed (we can the easily see that B is the common zero set of the invariants in $K[X]$ that are zero on A). In general, this is not true. For example, if $G = K^\times$ acts by $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ on $K^2\backslash\{0\}$, then $B = K^2\backslash(\{0\} \times K)$.

Lemma 2 *For almost every orbit of G on X to be closed, it is necessary and sufficient that the closure of B has empty interior.*

PROOF: Let T be an orbit in X. It is well-known that $\overline{T}\backslash T$ consists of orbits of dimension strictly less than that of T ([1, p. 98]). It follows that $\overline{T}\backslash T \subset A$. From this it follows that the closed orbits in X complementary to A in X are precisely the orbits complementary to B in X. Hence the condition of the lemma is sufficient. As the closed set A has empty interior, we find that it is also necessary. $\diamond$

Lemma 3 *Let G be an algebraic group that acts on two varieties X and Y, and let $\psi : X \to Y$ be a G-equivariant étale morphism.*

(1) *If almost every orbit in Y is closed, then almost every orbit in X is closed as well.*

(2) *If ψ is also surjective, the converse in (1) is true.*

PROOF: As ψ is étale, the inverse image of every closed orbit in Y is a finite union of closed orbits in X. From this, (1) follows immediately.

If ψ is also surjective, we show first that $B(Y) \subset \psi(B(X))$: Let Y be an orbit in $B(Y)$. The closure of T then contains a point $y \in A(Y)$. Let $x \in \psi^{-1}(y)$. As ψ is étale, we see that $x \in A(X)$. Let $T_1, \ldots, T_n$ denote the different orbits in $\psi^{-1}(T)$. If there exists a neighborhood V of x that does not intersect any of the T_i, the $\psi(V)$ is a neighborhood of y that does not intersect T, but this is absurd. Therefore, at least one of the T_i contains x in its closure, and thus is contained in $B(X)$. It follows that $T \subset \psi(B(X))$.

Assume now that almost every orbit in X is closed. The closure of $B(X)$ then has empty interior (Lemma 2). It is the same for $\psi(\overline{B(X)})$ and $\overline{\psi(\overline{B(X)})}$ (as ψ is étale and $\psi(\overline{B(X)})$ is contructible, [6, p. 97]). Consequently, the closure of $B(Y)$, which is contained in $\overline{\psi(\overline{B(X)})}$, also has empty interior. By virtue of Lemma 2 we conclude the second assertion of the lemma. $\diamond$

Return to the notations and hypothese of the previous paragraph. Let (φ, N) be an étale slice at $x \in X$, and let $V = V(\varphi, N)$ and $U = U(\varphi, N)$ be the open sets in $Y = \varphi^{-1}(N)$ and X that were introduced there.

Lemma 4 *If almost every orbit of G_x in N is closed, then almost every orbit of G in U is also closed.*

PROOF: The orbits of G in $G \times_{G_x} V$ are of the form $G \times_{G_x} T$, where T is an orbit of G_x in V; and $G \times_{G_x} T$ is closed in $G \times_{G_x} V$ if and only if T is closed in V. The lemma now follows immediately from Lemma 3. ◇

3 Orthogonalizable varieties

Let G be an algebraic group that acts on an algebraic variety X. We say that X is *(G-)orthogonalizable* if at every point in X, the tangent space admits a non-degenerate symmetric bilinear form that is invariant under the isotropy subgroup. If X is a G-module (that is, a vector space over K of finite dimension with a linear G-action), it is orthogonalizable if and only if it has a G-invariant non-degenerate symmetric bilinear form.

Lemma 5 *Suppose G is reductive and let M be a G-module and N a G-submodule of M. If M and N art orthogonalizable, then M/N is orthogonalizable as well.*

PROOF: (following [7, p. 144]) Let L be a G-invariant complement of N in M. We choose a G-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)_1$ on M, and one on N which we complete by 0 on L to a degenerate (unless $L = \mathbf{0}$) G-invariant symmetric bilinear form $(\cdot,\cdot)_2$ on M. On the "line" passing through $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ we can find a form $(\cdot,\cdot)$ that is non-degenerate on M and on N. The restriction of $(\cdot,\cdot)$ to the orthogonal space of N (with respect to $(\cdot,\cdot)$) is then also non-degenerate, and this G-module is isomorphic to M/N. ◇

We now fill in the details for the example in the introduction. Let H be a reductive group with Lie algebra $\mathfrak{h}$. The group H acts on $\mathfrak{h}$ by the adjoint representation. It is well-nown that $\mathfrak{h}$ is H-orthogonalizable. Let H_1 be a reductive subgroup of H with Lie algebra $\mathfrak{h}_1 \subset \mathfrak{h}$. The homogeneous space H/H_1 is an open affine variety [3]. The isotropy group of H at the point eH_1 is H_1, the tangent space at of H/H_1 at the point eH_1 can be identified with the H_1-module $\mathfrak{h}/\mathfrak{h}_1$. By Lemma 5, $\mathfrak{h}/\mathfrak{h}_1$ is H_1-orthogonalizable. It then follows that H/H_1 is H- and hence H_2-orthogonalizable for all subgroups H_2 of H.

4 Proof of the theorem

The proof is by induction on $\dim X$. If the dimension is 0, then there is nothing to prove.

Suppose now that $\dim X > 0$ and that the statement holds for all reductive groups acting on an orthogonalizable open affine variety of dimension less than $\dim X$.

Choose successively ($i = 1, 2, \ldots$) points $x_i \in X$ whose isotropy subgroups are reductive, and, at every point x_i, an étale slice (φ_i, N_i), by the following procedure: Given the points and their étale slices for $i < j$, take x_j in the complement of the union of the $U_i = U(\varphi_i, N_i)$ in such a way that the orbit $G(x_j)$ is closed (so that $G(x_j)$ is affine and hence G_{x_j} reductive [3]). As the topological space X is Noetherian, this constructions stops after a finite number of steps, when the U_i cover X.

By Lemma 5, the N_i are orthogonalizable, for $\mathrm{T}_{x_i} X$ and $\mathrm{T}_{x_i} G(x_i) \cong \mathfrak{g}/\mathfrak{g}_{x_i}$ are (we denote by $\mathfrak{g}$ and $\mathfrak{g}_{x_i}$ the Lie algebras of G and G_{x_i}). Choose a G_{x_i}-invariant non-degenerate symmetric bilinear form on N_i. The "spheres" of non-null rays with origin in N_i are then G_{x_i}-invariant smooth affine varieties, G_{x_i}-orthogonalizable and of dimension less than $\dim X$. By the induction hypthesis, almost all of their orbits are closed. It then follows immediately that almost all orbits of G_{x_i} in N_i are closed. By Lemma 4, this is the same as almost all orbits of G in U_i being closed.

Denote by U the disjoint union of the U_i. The inclusion of U_i in X defines a surjective étale morphism $U \to X$ that commutes with the G-action. It is clear that almost all orbits in U are closed. By Lemma 3, this is the same as almost all orbits in X being closed. $\diamond$

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Original: *Sur les orbites fermées des groupes algébriques reductifs*, Inventiones Mathematicae, 1972, 1-5

Translation by Wolfgang Globke, Version of February 24, 2017.