Tables of Prehomogeneous Modules and Étale Modules of Reductive Algebraic Groups

Compiled by Wolfgang Globke,
based on the results cited in the reference section.

Contents

1 Groups

In this thesis, we will mostly be concerned with subgroups of the **general linear group**,

$$\mathrm{GL}_n(\mathbb{k}) = \{g \in \mathrm{Mat}_n(\mathbb{k}) \mid \det(g) \neq 0\},$$

the group of invertible matrices. As GL_n can be considered as the complement of the closed set of singular matrices in $\mathbb{k}^{n^2}$, it is an affine variety, and as such, it is an algebraic group. Its Lie algebra is the set of $n \times n$-matrices, $\mathfrak{gl}_n$ with the matrix commutator as a Lie bracket.

Definition 1.1 A **linear algebraic group** G is an algebraic group which is a subgroup of GL_n.

Equivalently, linear algebraic groups are the subgroups of GL_n defined by certain polynomial equations.

1.1 SL_n

The **special linear group** is the group of unimodular matrices,

$$\mathrm{SL}_n(\mathbb{k}) = \{g \in \mathrm{GL}_n(\mathbb{k}) \mid \det(g) = 1\}$$

with Lie algebra

$$\mathfrak{sl}_n(\mathbb{k}) = \{X \in \mathfrak{gl}_n(\mathbb{k}) \mid \mathrm{trace}(X) = 0\}.$$

Its dimension is

$$\dim(\mathrm{SL}_n) = n^2 - 1.$$

This group is connected and simple (for $n \geq 2$) and its centre is a finite subgroup isomorphic to the group of n-th roots of unity in $\mathbb{k}$.

For $\mathbb{k} = \mathbb{R}$, the elements of SL_n can be interpreted geometrically as those linear transformations preserving volume and orientation.

1.2 Sp_n

Define the matrix J by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}.$$

The **symplectic group**[1)] is the group

$$\mathrm{Sp}_n(\mathbb{k}) = \{g \in \mathrm{GL}_{2n}(\mathbb{k}) \mid g^\top J g = J\}$$

with Lie algebra

$$\mathfrak{sp}_n(\mathbb{k}) = \{X \in \mathfrak{gl}_n(\mathbb{k}) \mid X^\top J + JX = 0\}.$$

Its dimension is

$$\dim(\mathrm{Sp}_n) = n(2n + 1).$$

By the condition $gJg^\top$ one easily sees that $\det(g) = \pm 1$, and even $\mathrm{Sp}_n \subset \mathrm{SL}_{2n}$ holds. Further, Sp_n is a simple and connected group.

1) Note that the notation Sp_{2n} instead of Sp_n is also used in the literature.

1.3 SO_n and Spin_n

The **orthogonal group** is

$$\mathrm{O}_n(\Bbbk) = \{g \in \mathrm{GL}_n(\Bbbk) \mid gg^\top = I_n\},$$

and its intersection with SL_n is the **special orthogonal group**

$$\mathrm{SO}_n(\Bbbk) = \{g \in \mathrm{O}_n(\Bbbk) \mid \det(g) = 1\}.$$

Both groups have the same Lie algebra

$$\mathfrak{o}_n(\Bbbk) = \mathfrak{so}_n(\Bbbk) = \{X \in \mathfrak{gl}_n(\Bbbk) \mid X^\top = -X\}.$$

They are of the same dimension,

$$\dim(\mathrm{O}_n) = \dim(\mathrm{SO}_n) = \frac{1}{2}n(n-1).$$

SO_n is connected, but O_n is not because

$$\mathrm{SO}_n = \mathrm{O}_n/\{\pm I_n\},$$

and $\mathrm{SO}_n = \mathrm{O}_n^\circ$. For $n \geq 3$, both groups are simple. But for $n = 2$, the group O_2 is abelian, hence not simple.

For $\Bbbk = \mathbb{R}$, the elements of O_n can be interpreted geometrically as the linear transformations preserving angles and lengths.

The definition can be generalised by requiring $gQg^\top = Q$ instead of $gg^\top = I_n$, where Q is a matrix defining a symmetric non-degenerate bilinear form.

Closely related to the orthogonal groups is the **spin group** $\mathrm{Spin}_n(\Bbbk)$, of which a detailed introduction can be found in chapter 20 of Fulton, Harris [2]. Here, we will just note that $\mathrm{Spin}_n/\{\pm 1\} \cong \mathrm{SO}_n$. In particular, the spin group has the same Lie algebra as O_n and SO_n, and it is essential in constructing some of the representations of this Lie algebra.

1.4 Exceptional Groups

Aside from the simple groups described above, there are five simple **exceptional groups**. These are the groups G_2, F_4, E_6, E_7 and E_8. They are rather complicated to describe in detail, so we will not bother to do this here, but give some references instead.

In the course of § 1 of Sato, Kimura [9], a description of these exceptional groups is given. Chapter 22 of Fulton, Harris [2] is dedicated to the construction of their Lie algebras from the root data.

1.5 Other Groups

Some other groups which are not simple appear in the course of this thesis.

First, the **additive group** G_m^+ of dimension m which can be considered as the vector space $\Bbbk^m$ with its addition as a group operation. In this thesis, this group arises as a semidirect factor of generic isotropy subgroups of prehomogeneous modules, see chapter 2, and in this context it is often written as $G^+_{m(n-m)}$, which indicates that it appears as a group of matrices of the form

$$\begin{pmatrix} I_{n-m} & 0 \\ A & I_m \end{pmatrix},$$

with $A \in \mathrm{Mat}_{m,n-m}$. Under multiplication, these matrices behave just like the additive group.

Next, there is the n-dimensional **multiplicative group** $(\Bbbk^\times)^n$, with componentwise multiplication in $\Bbbk^\times$. This group is identical to GL_1^n, and we shall use the latter notation most of the time.

A matrix group is **unipotent** if $(I_n - g)^k = 0$ holds for some $k \in \mathbb{N}$ and any element g. It can be shown that any unipotent group is isomorphic to a closed subgroup of the group of upper triangular matrices with 1 on the diagonal,

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Unipotent groups appear as semidirect factors of generic isotropy subgroups of prehomogeneous modules. Be warned though that in many cases these subgroups appear in a non-obvious representation, so see the cases in § 5 of Sato, Kimura [9] for the respective appearance of these groups. To be consistent with the notation of Sato, Kimura [9], we let Un_n denote a unipotent group of dimension n, but *not* the group of unipotent $n \times n$-matrices, which would be the more common usage.

2 Classification of Prehomogeneous Modules

2.1 Irreducible Reduced Prehomogeneous Modules

The irreducible and reduced prehomogeneous modules were classified by Sato and Kimura, thus we will label each class by SK n, where n is the number given to the class in § 7 of the the original work by Sato and Kimura [9]. Along with each module, we will state the connected component of the generic isotropy group, denoted by G_v°, and in some cases the irreducible relative invariant, denoted by f.

Let G be a reductive group and (G, ϱ, V) an irreducible and reduced prehomogeneous module. Then it is equivalent to one of the following prehomogeneous modules:

SK I Regular irreducible reduced prehomogeneous modules.

1. $(G \times \mathrm{GL}_m, \varrho \otimes \omega_1, V^m \otimes \Bbbk^m)$,
   where $\varrho : G \to \mathrm{GL}(V^m)$ is an m-dimensional irreducible representation of a connected semisimple algebraic group G (or $G = \{1\}$ and $m = 1$). We have $G_v^\circ \cong G$ and $f(x) = \det(x)$ for $x \in \mathrm{Mat}_m \cong V^m \otimes \Bbbk^m$, $\deg(f) = m$.
2. $(\mathrm{GL}_n, 2\omega_1, \mathrm{Sym}^2\Bbbk^n)$ for $n \geq 2$.
   We have $G_v^\circ \cong \mathrm{SO}_n$ and $f(x) = \det(x)$ for $x \in \{A \in \mathrm{Mat}_n \mid A^\top = A\} \cong \mathrm{Sym}^2\Bbbk^n$, $\deg(f) = n$.
3. $(\mathrm{GL}_{2n}, \omega_2, \bigwedge^2 \Bbbk^{2n})$ for $n \geq 3$.
   We have $G_v^\circ \cong \mathrm{Sp}_n$ and $f(x) = \mathrm{Pf}(x)$ for $x \in \{A \in \mathrm{Mat}_{2n} \mid A^\top = -A\} \cong \bigwedge^2 \Bbbk^{2n}$, $\deg(f) = n$.
4. $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\Bbbk^2)$.
   We have $G_v^\circ \cong \{1\}$ and $f(a) = a_2^2a_3^2 + 18a_1a_2a_3a_4 - 4a_1a_3^3 - 4a_2^3a_4 - 27a_1^2a_4^2$ for $a = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \in \mathrm{Sym}^3\Bbbk^2$ (so f is the discriminant of a binary cubic form $a(x, y)$).
5. $(\mathrm{GL}_6, \omega_3, \bigwedge^3 \Bbbk^6)$.
   We have $G_v^\circ \cong \mathrm{SL}_3 \times \mathrm{SL}_3$ and $f(x) = (x_0y_0 - \mathrm{trace}(XY))^2 + 4x_0\det(Y) + 4y_0\det(X) - 4\sum_{i,j}\det(X_{ij})\det(Y_{ji})$ (see § 5, p. 83 in [9] for a definition), $\deg(f) = 4$.
6. $(\mathrm{GL}_7, \omega_3, \bigwedge^3 \Bbbk^7)$.
   We have $G_v^\circ \cong \mathrm{G}_2$ and $\deg(f) = 7$.
7. $(\mathrm{GL}_8, \omega_3, \bigwedge^3 \Bbbk^2)$.
   We have $G_v^\circ \cong \mathrm{SL}_3$ and $\deg(f) = 16$.
8. $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2)$.
   We have $G_v^\circ \cong \{1\}$ and $f(A, B) = \mathrm{dis}(\det(xA + yB))$ for $(A, B) \in \{(X, Y) \mid X, Y \in \mathrm{Mat}_3, X^\top = X, Y^\top = Y\} \cong \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2$, $\deg(f) = 12$.
9. $(\mathrm{SL}_6 \times \mathrm{GL}_2, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^6 \otimes \Bbbk^2)$.
   We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and $f(A, B) = \mathrm{dis}(\mathrm{Pf}(xA + yB))$ for $(A, B) \in \{(X, Y) \mid X, Y \in \mathrm{Mat}_6, X^\top = -X, Y^\top = -Y\} \cong \bigwedge^2 \Bbbk^6 \otimes \Bbbk^2$, $\deg(f) = 12$.

10. $(\mathrm{SL}_5 \times \mathrm{GL}_3, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^3)$.
We have $G_v^\circ \cong \mathrm{SL}_2$ and $\deg(f) = 15$.

11. $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^4)$.
We have $G_v^\circ \cong \{1\}$ and $\deg(f) = 40$.

12. $(\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_2, \omega_1 \otimes \omega_1 \otimes \omega_1, \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^2)$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{GL}_1$ and $f(A, B) = \mathrm{dis}(\det(xA + yB))$ for $(A, B) \in \mathrm{Mat}_3 \oplus \mathrm{Mat}_3 \cong \Bbbk^3 \otimes \Bbbk^3 \otimes \Bbbk^2$, $\deg(f) = 12$.

13. $(\mathrm{Sp}_n \times \mathrm{GL}_{2m}, \omega_1 \otimes \omega_1, \Bbbk^{2n} \otimes \Bbbk^{2m})$ for $n \geq 2m \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_m \times \mathrm{Sp}_{n-m}$ and $f(X) = \mathrm{Pf}(X^\top J X)$ for $X \in \mathrm{Mat}_{2n,2m}$, $\deg(f) = 2m$.

14. $(\mathrm{GL}_1 \times \mathrm{Sp}_3, \mu \otimes \omega_3, \Bbbk \otimes V^{14})$.
We have $G_v^\circ \cong \mathrm{SL}_3$ and $\deg(f) = 4$, where f is given by the restriction of the relative invariant of SK I-5.

15. $(\mathrm{SO}_n \times \mathrm{GL}_m, \omega_1 \otimes \omega_1, \Bbbk^n \otimes \Bbbk^m)$ for $n \geq 3$, $\frac{1}{2}n \geq m \geq 1$.
We have $G_v^\circ \cong \mathrm{SO}_m \times \mathrm{SO}_{n-m}$ and $f(X) = \det(X^\top Q X)$ for $X \in \mathrm{Mat}_{n,m} \cong \Bbbk^n \otimes \Bbbk^m$, $\deg(f) = 2m$, where $Q = g^\top Q g$ for $g \in \mathrm{SO}_n$.

16. $(\mathrm{GL}_1 \times \mathrm{Spin}_7, \mu \otimes \mathrm{spinrep}, \Bbbk \otimes V^8)$.
We have $G_v^\circ \cong \mathrm{G}_2$ and $\deg(f) = 2$, where f is the relative invariant of SK I-15 for $m = 1$, $n = 8$.

17. $(\mathrm{GL}_2 \times \mathrm{Spin}_7, \omega_1 \otimes \mathrm{spinrep}, \Bbbk^2 \otimes V^8)$.
We have $G_v^\circ \cong \mathrm{SO}_2 \times \mathrm{SL}_3$ and $\deg(f) = 4$, where f is the relative invariant of SK I-15 for $m = 2$, $n = 8$.

18. $(\mathrm{GL}_3 \times \mathrm{Spin}_7, \omega_1 \otimes \mathrm{spinrep}, \Bbbk^3 \otimes V^8)$.
We have $G_v^\circ \cong \mathrm{SO}_3 \times \mathrm{SL}_2$ and $\deg(f) = 6$, where f is the relative invariant of SK I-15 for $m = 3$, $n = 8$.

19. $(\mathrm{GL}_1 \times \mathrm{Spin}_9, \mu \otimes \mathrm{spinrep}, \Bbbk \otimes V^{16})$.
We have $G_v^\circ \cong \mathrm{Spin}_7$ and $\deg(f) = 2$.

20. $(\mathrm{GL}_2 \times \mathrm{Spin}_{10}, \omega_1 \otimes \mathrm{halfspinrep}, \Bbbk^2 \otimes V^{16})$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{G}_2$ and $\deg(f) = 4$.

21. $(\mathrm{GL}_3 \times \mathrm{Spin}_{10}, \omega_1 \otimes \mathrm{halfspinrep}, \Bbbk^2 \otimes V^{16})$.
We have $G_v^\circ \cong \mathrm{SO}_3 \times \mathrm{SL}_2$ and $\deg(f) = 12$.

22. $(\mathrm{GL}_1 \times \mathrm{Spin}_{11}, \mu \otimes \mathrm{spinrep}, \Bbbk \otimes V^{32})$.
We have $G_v^\circ \cong \mathrm{SL}_5$ and $\deg(f) = 4$.

23. $(\mathrm{GL}_1 \times \mathrm{Spin}_{12}, \mu \otimes \mathrm{halfspinrep}, \Bbbk \otimes V^{32})$.
We have $G_v^\circ \cong \mathrm{SL}_6$ and $\deg(f) = 4$.

24. $(\mathrm{GL}_1 \times \mathrm{Spin}_{14}, \mu \otimes \mathrm{halfspinrep}, \Bbbk \otimes V^{64})$.
We have $G_v^\circ \cong \mathrm{G}_2 \times \mathrm{G}_2$ and $\deg(f) = 8$.

25. $(\mathrm{GL}_1 \times \mathrm{G}_2, \mu \otimes \omega_2, \Bbbk \otimes V^7)$.
We have $G_v^\circ \cong \mathrm{SL}_3$ and $\deg(f) = 2$, where f is the relative invariant of SK I-15 for $m = 1$, $n = 7$.

26. $(\mathrm{GL}_2 \times \mathrm{G}_2, \omega_1 \otimes \omega_2, \Bbbk^2 \otimes V^7)$.
We have $G_v^\circ \cong \mathrm{GL}_2$ and $\deg(f) = 4$, where f is the relative invariant of SK I-15 for $m = 2, n = 7$.

27. $(\mathrm{GL}_1 \times \mathrm{E}_6, \mu \otimes \omega_1, \Bbbk \otimes V^{27})$.
We have $G_v^\circ \cong \mathrm{F}_4$ and $\deg(f) = 4$.

28. $(\mathrm{GL}_2 \times \mathrm{E}_6, \omega_1 \otimes \omega_1, \Bbbk^2 \otimes V^{27})$.
We have $G_v^\circ \cong \mathrm{SO}_8$ and $\deg(f) = 12$.

29. $(\mathrm{GL}_1 \times \mathrm{E}_7, \mu \otimes \omega_6, \Bbbk \otimes V^{56})$.
We have $G_v^\circ \cong \mathrm{E}_6$ and $\deg(f) = 4$.

SK II Non-regular irreducible reduced prehomogeneous modules with non-constant relative invariant.

1. $(\mathrm{GL}_1 \times \mathrm{Sp}_n \times \mathrm{SO}_3, \mu \otimes \omega_1 \otimes \omega_1, \Bbbk \otimes \Bbbk^{2n} \otimes \Bbbk^3)$.
We have $G_v^\circ \cong (\mathrm{Sp}_{n-2} \times \mathrm{SO}_2) \cdot \mathrm{Un}_{2n-3}$ and $f(X) = \mathrm{trace}(X^\top JXQ)^2$ for $X \in \mathrm{Mat}_{2n,3} \cong \Bbbk \otimes \Bbbk^{2n} \otimes \Bbbk^3$.

SK III Non-regular irreducible reduced prehomogeneous modules without non-constant relative invariants.

1. $(G \times \mathrm{GL}_m, \varrho \otimes \omega_1, V^n \otimes \Bbbk^m)$,
where $\varrho : G \to \mathrm{GL}(V^n)$ is an n-dimensional irreducible representation of a semisimple algebraic group $G(\neq \mathrm{SL}_n)$ with $m > n \geq 3$. We have $G_v^\circ \cong (G \times \mathrm{GL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$. The module $(G \times \mathrm{SL}_m, \varrho \otimes \omega_1)$ is prehomogeneous with $G_v^\circ \cong (G \times \mathrm{SL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$.

2. $(\mathrm{SL}_n \times \mathrm{GL}_m, \omega_1 \otimes \omega_1, \Bbbk^n \otimes \Bbbk^m)$ for $\frac{1}{2}m \geq n \geq 1$.
We have $G_v^\circ \cong (\mathrm{SL}_n \times \mathrm{GL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$. The module $(\mathrm{SL}_n \times \mathrm{SL}_m, \omega_1 \otimes \omega_1)$ is prehomogeneous with $G_v^\circ \cong (\mathrm{SL}_n \times \mathrm{SL}_{m-n}) \cdot \mathrm{G}^+_{n(m-n)}$.

3. $(\mathrm{GL}_{2n+1}, \omega_2, \bigwedge^2 \Bbbk^{2n+1})$ for $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{Sp}_n \times \mathrm{GL}_1) \cdot \mathrm{G}^+_{2n}$. The module $(\mathrm{SL}_{2n+1}, \omega_2)$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_n \cdot \mathrm{G}^+_{2n}$.

4. $(\mathrm{GL}_2 \times \mathrm{SL}_{2n+1}, \omega_1 \otimes \omega_2, \Bbbk^2 \otimes \bigwedge^2 \Bbbk^{2n+1})$ for $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SL}_2) \cdot \mathrm{G}^+_{2n}$ (see lemma 1.4 in Kimura et al. [5]). The module $(\mathrm{SL}_2 \times \mathrm{SL}_{2n+1}, \omega_1 \otimes \omega_2)$ is prehomogeneous with $G_v^\circ \cong \mathrm{SL}_2 \cdot \mathrm{G}^+_{2n}$.

5. $(\mathrm{Sp}_n \times \mathrm{GL}_{2m+1}, \omega_1 \otimes \omega_1, \Bbbk^{2n} \otimes \Bbbk^{2m+1})$ for $n > 2m + 1 \geq 1$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m}) \cdot \mathrm{Un}_{2n-1}$. The module $(\mathrm{Sp}_n \times \mathrm{SL}_{2m+1}, \omega_1 \otimes \omega_1)$ is prehomogeneous with $G_v^\circ \cong (\mathrm{Sp}_m \times \mathrm{Sp}_{n-m}) \cdot \mathrm{Un}_{2n-1}$.

6. $(\mathrm{GL}_1 \times \mathrm{Spin}_{10}, \mu \otimes \mathrm{halfspinrep}, \Bbbk \otimes V^{16})$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Spin}_7) \cdot \mathrm{G}^+_8$. The module $(\mathrm{Spin}_{10}, \mathrm{halfspinrep})$ is prehomogeneous with $G_v^\circ \cong \mathrm{Spin}_7 \cdot \mathrm{G}^+_8$.

2.2 Non-Irreducible Simple Prehomogeneous Modules

The simple prehomogeneous, including the non-irreducible ones, were classified by Kimura, thus we will label them by Ks n, where n is the number of the module in § 3 of Kimura's article [4].

In this section, it is understood that each representation ϱ_i of the simple group is composed with a scalar multiplication μ of GL_1^k. We shall simply write ϱ_i instead of $\mu \otimes \varrho_i$. In some cases, a module $V_1 \oplus \ldots \oplus V_k$ will be prehomogeneous even with fewer than k scalar multiplications, in which case we will state this fact explicitely. We shall also state the connected component G_v° of the generic isotropy subgroup and the relative invariants $f_1, \ldots, f_l$ where they exist.

Let $G = \mathrm{GL}_1^k \times G_\mathrm{s}$ be a reductive group, where G_s is a simple algebraic group, let $(\varrho_1, V_1), \ldots, (\varrho_k, V_k)$ be irreducible G_s-modules and $V = V_1 \oplus \ldots \oplus V_k$ a G_s-module with representation $\varrho = \varrho_1 \oplus \ldots \oplus \varrho_k$. Then (G, ϱ, V) is equivalent to one of the following:

Ks I Regular non-irreducible simple prehomogeneous modules.

1. $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, \omega_1 \oplus \omega_1^*, \Bbbk^n \oplus \Bbbk^{n*})$ for $n \geq 3$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{SL}_{n-1}$ and $f_1(x,y) = \langle x|y\rangle$, where $(x,y) \in \Bbbk^n \oplus \Bbbk^{n*}$ and $\langle \cdot \mid \cdot \rangle$ is the dual pairing. The module $(\mathrm{GL}_1 \times \mathrm{SL}_n, (\mu \otimes \omega_1) \oplus \omega_1^*)$ is prehomogeneous with $G_v^\circ = \mathrm{SL}_{n-1}$.
2. $(\mathrm{GL}_1^n \times \mathrm{SL}_n, \omega_1^{\oplus n}, (\Bbbk^n)^{\oplus n})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1^{n-1}$ and $f_1(X) = \det(X)$ for $X \in \mathrm{Mat}_n \cong (\Bbbk^n)^{\oplus n}$. The module $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n})$ is prehomogeneous with $G_v^\circ = \{1\}$.
3. $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\Bbbk^n)^{\oplus n+1})$ for $n \geq 2$.
We have $G_v^\circ \cong \{1\}$ and $f_i(X) = \det(x_1, \ldots, \not{x_i}, \ldots, x_{n+1})$ for $X = (x_1, \ldots, x_{n+1}) \in \mathrm{Mat}_{n,n+1} \cong (\Bbbk^n)^{\oplus n+1}$.
4. $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*})$ for $n \geq 3$.
We have $G_v^\circ \cong \{1\}$ and $f_1(X) = \langle x_1|y\rangle, \ldots, f_n(X) = \langle x_n|y\rangle$, $f_{n+1}(X) = \det(x_1, \ldots, x_n)$ for $X = (x_1, \ldots, x_n, y) \in (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*}$.
5. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ and $f_1(X,y,z) = \mathrm{Pf}(X)$, $f_2(X,y,z) = y^\top X^\# z$, where $(X,y,z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n}$ and $X^\#$ is the cofactor matrix of X. The module $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, (\mu \otimes \omega_2) \oplus (\mu \otimes (\omega_1 \oplus \omega_1)))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1}$.
6. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ and $f_1(X,y,z) = \mathrm{Pf}(X)$, $f_2(X,y,z) = \langle y|z\rangle$, where $(X,y,z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*}$. The module $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, (\mu \otimes \omega_2) \oplus (\mu \otimes (\omega_1 \oplus \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1}$.
7. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*})$ for $n \geq 3$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$ and $f_1(X,y,z) = \mathrm{Pf}(X)$, $f_2(X,y,z) = y^\top X z$, where $(X,y,z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*}$. The module $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, (\mu \otimes \omega_2) \oplus (\mu \otimes (\omega_1^* \oplus \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1}$.

8. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_n$. The module $(\mathrm{GL}_1 \times \mathrm{Sp}_{2n+1}, \mu \otimes (\omega_2 \oplus \omega_1))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_n$, see p. 94 in [4] for the relative invariant.
9. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-1}$, see p. 94 in [4] for the relative invariants.
10. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-1}$ and $f_1(X) = \mathrm{Pf}\begin{pmatrix} A & x \\ x^\top & 0 \end{pmatrix}$, $f_2(X) = \langle x|y\rangle$, $f_3(X) = \langle x|z\rangle$, $f_4 = y^\top A z$ for $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*}$.
11. $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^n)$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{SO}_{n-1}$ and $f_1(X) = \det(A)$, $f_2(X) = x^\top A^\# x$ for $X = (A, x) \in \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^n$.
12. $(\mathrm{GL}_1^2 \times \mathrm{SL}_n, 2\omega_1 \oplus \omega_1^*, \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^{n*})$ for $n \geq 3$.
We have $G_v^\circ \cong \mathrm{SO}_{n-1}$ and $f_1(X) = \det(A)$, $f_2(X) = x^\top A x$ for $X = (A, x) \in \mathrm{Sym}^2\Bbbk^n \oplus \Bbbk^n$.
13. $(\mathrm{GL}_1^2 \times \mathrm{SL}_7, \omega_3 \oplus \omega_1, \bigwedge^3 \Bbbk^7 \oplus \Bbbk^7)$.
We have $G_v^\circ \cong \mathrm{SL}_3$, see p. 96 in [4] for the relative invariants.
14. $(\mathrm{GL}_1^2 \times \mathrm{SL}_7, \omega_3 \oplus \omega_1^*, \bigwedge^3 \Bbbk^7 \oplus \Bbbk^{7*})$.
We have $G_v^\circ \cong \mathrm{SL}_3$, see p. 96 in [4] for the relative invariants.
15. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_8, \text{spinrep} \oplus \text{halfspinrep}, V^8 \oplus V^8)$.
We have $G_v^\circ \cong \mathrm{G}_2$ and two quadratic invariants $f_1(x, y) = q_1(x)$, $f_2(x, y) = q_2(y)$ for $(x, y) \in V^8 \oplus V^8$.
16. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7, \text{vecrep} \oplus \text{spinrep}, V^7 \oplus V^8)$.
We have $G_v^\circ \cong \mathrm{SL}_3$ and two quadratic invariants $f_1(x, y) = q_1(x)$, $f_2(x, y) = q_2(y)$ for $(x, y) \in V^7 \oplus V^8$.
17. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10}, \text{halfspinrep}_{\text{even}} \oplus \text{halfspinrep}_{\text{even}}, V^{16} \oplus V^{16})$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{G}_2$, see p. 96 in [4] for the relative invariants. The module $(\mathrm{GL}_1 \times \mathrm{Spin}_{10}, \mu \otimes (\text{halfspinrep}_{\text{even}} \oplus \text{halfspinrep}_{\text{even}}))$ is prehomogeneous with $G_v^\circ \cong \mathrm{G}_2$.
18. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10}, \text{vecrep} \oplus \text{halfspinrep}, V^{10} \oplus V^{16})$.
We have $G_v^\circ \cong \mathrm{Spin}_7$, see p. 97 in [4] for the relative invariants.
19. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{12}, \text{vecrep} \oplus \text{halfspinrep}, V^{12} \oplus V^{32})$.
We have $G_v^\circ \cong \mathrm{SL}_5$, see p. 97 in [4] for the relative invariants.
20. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_n, \omega_1 \oplus \omega_1, \Bbbk^{2n} \oplus \Bbbk^{2n})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{Sp}_{n-1}$, see p. 97 in [4] for the relative invariant. The module $(\mathrm{GL}_1 \times \mathrm{Sp}_n, \mu \otimes (\omega_1 \oplus \omega_1))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1}$.
21. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_3, \omega_3 \oplus \omega_1, V^{14} \oplus \Bbbk^6)$.
We have $G_v^\circ \cong \mathrm{SL}_2$, see p. 97 in [4] for the relative invariants.

Ks II Non-regular non-irreducible simple prehomogeneous modules.

1. $(\mathrm{GL}_1^k \times \mathrm{SL}_n, \omega_1^{\oplus k}, (\Bbbk^n)^{\oplus k})$ for $2 \le k \le n-1$.
   We have $G_v^\circ \cong (\mathrm{GL}_1^k \times \mathrm{SL}_{n-k}) \cdot \mathrm{G}^+_{k(n-k)}$. The module $(\mathrm{SL}_n, \omega_1^{\oplus k})$ is prehomogeneous with $G_v^\circ \cong \mathrm{SL}_{n-k} \cdot \mathrm{G}^+_{k(n-k)}$.
2. $(\mathrm{GL}_1^k \times \mathrm{SL}_n, \omega_1^{\oplus k-1} \oplus \omega_1^*, (\Bbbk^n)^{\oplus k-1} \oplus \Bbbk^{n*})$ for $3 \le k \le n$.
   We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SL}_{n-k+1}) \cdot \mathrm{G}^+_{(n-k+1)(k-2)}$ and $f_1(X) = \langle x_1 | y \rangle, \dots, f_{k-1}(X) = \langle x_{k-1} | y \rangle$ for $X = (x_1, \dots, x_{k-1}, y) \in (\Bbbk^n)^{\oplus k-1} \oplus \Bbbk^{n*}$. The module $(\mathrm{GL}_1^{k-1} \times \mathrm{SL}_n, (\mu \otimes \omega_1^{\oplus k-1}) \oplus \omega_1^*)$ is prehomogeneous with $G_v^\circ \cong \mathrm{SL}_{n-k+1} \cdot \mathrm{G}^+_{(n-k+1)(k-2)}$.
3. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_2, \bigwedge^2 \Bbbk^{2n+1} \oplus \bigwedge^2 \Bbbk^{2n+1})$ for $n \ge 2$.
   We have $G_v^\circ \cong \mathrm{GL}_1^2 \cdot \mathrm{G}^+_{2n}$. The module $(\mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_2)$ is prehomogeneous with $G_v^\circ \cong \mathrm{G}^+_{2n}$.
4. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n})$ for $n \ge 2$.
   We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{n-1}$ and $f_1(X) = \mathrm{Pf}(X)$ where $X \in \bigwedge^2 \Bbbk^{2n}$. The module $(\mathrm{GL}_1 \times \mathrm{SL}_{2n}, \mu \otimes (\omega_2 \oplus \omega_1))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{n-1}$.
5. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*})$ for $n \ge 3$.
   We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{n-1}$ and $f_1(X) = \mathrm{Pf}(X)$ where $X \in \bigwedge^2 \Bbbk^{2n}$. The module $(\mathrm{GL}_1 \times \mathrm{SL}_{2n}, \mu \otimes (\omega_2 \oplus \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{n-1}$.
6. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n})$ for $n \ge 2$.
   We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ and $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = x^\top A^\# y$, $f_3(X) = y^\top A^\# z$, $f_4(X) = z^\top A x$ for $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n}$.
7. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*})$ for $n \ge 2$.
   We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ and $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = x^\top A^\# y$, $f_3(X) = \langle x | z \rangle$, $f_4(X) = \langle y | z \rangle$ for $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*}$.
8. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*})$ for $n \ge 3$.
   We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ and $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = \langle x | y \rangle$, $f_3(X) = \langle x | z \rangle$, $f_4(X) = y^\top A z$ for $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*}$.
9. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n}, \omega_2 \oplus \omega_1^* \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*})$ for $n \ge 3$.
   We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$ and $f_1(X) = \mathrm{Pf}(A)$, $f_2(X) = x^\top A y$, $f_3(X) = y^\top A z$, $f_4(X) = z^\top A x$ for $X = (A, x, y, z) \in \bigwedge^2 \Bbbk^{2n} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*} \oplus \Bbbk^{2n*}$.
10. $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*})$ for $n \ge 2$.
    We have $G_v^\circ \cong (\mathrm{GL}_1^2 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{4n-2}$. The module $(\mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^*)$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{4n-2}$.
11. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ for $n \ge 2$.
    We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{2n-1}$, see p. 99 in [4] for the relative invariants. The module $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, (\mu \otimes (\omega_2 \oplus \omega_1)) \oplus (\mu \otimes \omega_1))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{2n-1}$.
12. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1 \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*})$ for $n \ge 2$.
    We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{2n-1}$, see p. 99 in [4] for the relative

invariants. The module $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1}, (\mu \otimes (\omega_2 \oplus \omega_1)) \oplus (\mu \otimes \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{2n-1}$.

13. $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*})$ for $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-1}) \cdot \mathrm{Un}_{2n-1}$ and $f_1(X) = x^\top A y$ for $X = (A, x, y) \in \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1*} \oplus \Bbbk^{2n+1*}$. The module $(\mathrm{GL}_1 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus (\mu \otimes (\omega_1^* \oplus \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-1} \cdot \mathrm{Un}_{2n-2}$.

14. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus \omega_1^* \oplus \omega_1^* \oplus \omega_1^*, \bigwedge^2 \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1} \oplus \Bbbk^{2n+1})$ for $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{Sp}_{n-2}) \cdot \mathrm{Un}_{4n-6}$ and $f_1(X) = x^\top A y$, $f_2(X) = y^\top A z$, $f_3(X) = z^\top A x$. The module $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1}, \omega_2 \oplus (\mu \otimes \omega_1^*) \oplus (\mu \otimes \omega_1^*) \oplus (\mu \otimes \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{4n-6}$.

15. $(\mathrm{GL}_1^2 \times \mathrm{SL}_6, \omega_3 \oplus \omega_1, \bigwedge^3 \Bbbk^6 \oplus \Bbbk^6)$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{SL}_2) \cdot \mathrm{G}_4^+$, see p. 100 in [4] for the relative invariant. The module $(\mathrm{GL}_1 \times \mathrm{SL}_6, \mu \otimes (\omega_3 \oplus \omega_1))$ is prehomogeneous with $G_v^\circ \cong (\mathrm{SL}_2 \times \mathrm{SL}_2) \cdot \mathrm{G}_4^+$.

16. $(\mathrm{GL}_1^3 \times \mathrm{SL}_6, \omega_3 \oplus \omega_1 \oplus \omega_1, \bigwedge^3 \Bbbk^6 \oplus \Bbbk^6 \oplus \Bbbk^6)$.
We have $G_v^\circ \cong \mathrm{GL}_1^2 \cdot \mathrm{G}_4^+$, see p. 100 in [4] for the relative invariant. The module $(\mathrm{GL}_1 \times \mathrm{SL}_6, \mu \otimes (\omega_3 \oplus \omega_1 \oplus \omega_1))$ is prehomogeneous with $G_v^\circ \cong \mathrm{G}_4^+$.

17. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_n, \omega_1 \oplus \omega_1 \oplus \omega_1, \Bbbk^{2n} \oplus \Bbbk^{2n} \oplus \Bbbk^{2n})$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{Sp}_{n-2} \cdot \mathrm{Un}_{2n-3}$, see p. 100 in [4] for the relative invariants.

18. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2, \omega_2 \oplus \omega_1, V^5 \oplus \Bbbk^4)$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{Un}_2$, see p. 100 in [4] for the relative invariant. The module $(\mathrm{GL}_1 \times \mathrm{Sp}_2, (\mu \otimes \omega_2) \oplus \omega_1)$ is prehomogeneous with $G_v^\circ \cong \mathrm{Un}_2$.

19. $(\mathrm{GL}_1^3 \times \mathrm{SL}_5, \omega_2 \oplus \omega_2 \oplus \omega_1^*, \bigwedge^2 \Bbbk^5 \oplus \bigwedge^2 \Bbbk^5 \oplus \Bbbk^5)$.
See proposition 1.1 in [5].

2.3 2-Simple Prehomogeneous Modules of Type I

In this and the following chapter we shall give a classification of the non-irreducible 2-simple prehomogeneous modules, i.e. modules of the form

$$\Big(\mathrm{GL}_1^l \times G_1 \times G_2,\\ (\varrho_1 \otimes \tilde{\varrho}_1) \oplus \ldots \oplus (\varrho_k \otimes \tilde{\varrho}_k) \;\oplus\; (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \;\oplus\; (1 \otimes \tau_1) \oplus \ldots \oplus (1 \otimes \tau_t),\\ V_1 \oplus \ldots \oplus V_l\Big),$$

where G_1 and G_2 are simple algebraic groups, $l = k + s + t$, and the ϱ_i, σ_j (resp. $\tilde{\varrho}_i$, τ_j) are irreducible representations of G_1 (resp. G_2). As in the previous chapter, it is understood that each of these representations is composed with a scalar multiplication of GL_1^k. First, we give the classification of the type I-modules, i.e. at least one of the modules $(\mathrm{GL}_1 \times G_1 \times G_2, \varrho_i \otimes \tilde{\varrho}_i)$ is a non-trivial prehomogeneous module. These were classified by Kimura et al. [5], thus we shall refer to them as KI n, where n is the number of the module in § 3 of [5]. We shall state the non-irreducible modules only, as the irreducible ones already appear in the table

SK or as castling transformas of those (see also theorem 1.5 in [5]). In the next chapter, we shall classify the remaining 2-simple modules of type II.

Let (G, ϱ, V) be a 2-simple prehomogeneous module of type I. Then it is equivalent to one of the following:

KI I Regular 2-simple prehomogeneous modules of type I.

1. $(GL_1^2 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.
We have $G_v^\circ \cong \{1\}$.
2. $(GL_1^3 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong GL_1$. The module $(GL_1^2 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1))$ is prehomogeneous with $G_v^\circ \cong \{1\}$.
3. $(GL_1^2 \times SL_4 \times SL_3, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong SO_3$.
4. $(GL_1^3 \times SL_4 \times SL_3, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong SO_2$.
5. $(GL_1^3 \times SL_4 \times SL_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong SO_2$.
6. $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1))$.
We have $G_v^\circ \cong \{1\}$.
7. $(GL_1^2 \times SL_5 \times SL_3, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong SO_2$.
8. $(GL_1^2 \times SL_5 \times SL_8, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong SO_2$.
9. $(GL_1^2 \times SL_5 \times SL_9, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong GL_1 \times SL_2 \times SL_2$. The module $(GL_1 \times SL_5 \times SL_9, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong SL_2 \times SL_2$.
10. $(GL_1^3 \times Sp_n \times SL_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong GL_1 \times Sp_{n-m} \times Sp_{m-1}$. The module $(GL_1^2 \times Sp_n \times Sp_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ is prehomogeneous with $G_v^\circ \cong Sp_{n-m} \times Sp_{m-1}$.
11. $(GL_1^2 \times Sp_n \times SL_2, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$.
We have $G_v^\circ \cong Sp_{n-1} \times SO_2$.
12. $(GL_1^2 \times Sp_n \times SL_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 3\omega_1))$.
We have $G_v^\circ \cong Sp_{n-1}$.
13. $(GL_1^3 \times Sp_n \times SL_2, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_2) \oplus (1 \otimes \omega_1))$. We have $G_v^\circ \cong Sp_{n-1}$.
14. $(GL_1^2 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong GL_1 \times Sp_m \times Sp_{n-m-1}$. The module $(GL_1 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$ is prehomogeneous with $G_v^\circ \cong Sp_m \times Sp_{n-m-1}$.
15. $(GL_1^4 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes (\omega_1 \oplus \omega_1)^{(*)}))$.
We have $G_v^\circ \cong Sp_{m-1} \times Sp_{n-m-1}$.

16. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{GL}_1$. The module $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*))$ is prehomogeneous with $G_v^\circ \cong \{1\}$.
17. $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SO}_2$.
18. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \{1\}$.
19. $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \{1\}$.
20. $(\mathrm{GL}_1^2 \times \mathrm{SO}_n \times \mathrm{SL}_m, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SO}_{m-1} \times \mathrm{SO}_{n-m}$.
21. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_2, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.
22. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_3, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.
23. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_6, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.
24. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_7, (\mathrm{spinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.
25. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_7 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{spinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{GL}_2$.
26. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_7 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{spinrep} \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_2$.
27. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_7 \times \mathrm{SL}_6, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{spinrep} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2$.
28. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_8 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SL}_3 \times \mathrm{SO}_2$.
29. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_8 \times \mathrm{SL}_3, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_3$.
30. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_2, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.
31. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_3, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.
32. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_6, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.
33. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_8 \times \mathrm{SL}_7, (\mathrm{vecrep} \otimes \omega_1) \oplus (\mathrm{halfspinrep} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_3$.
34. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\mathrm{halfspinrep} \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2 \times \mathrm{SO}_3$.

35. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes 3\omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

36. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{G}_2$. The module $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ is prehomogeneous with $G_v^\circ \cong \mathrm{G}_2$.

37. $(\mathrm{GL}_1^3 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes 2\omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

38. $(\mathrm{GL}_1^4 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

39. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_3, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

40. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_{14}, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SO}_2$.

41. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_{15}, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{GL}_1 \times \mathrm{SL}_4$. The module $(\mathrm{GL}_1 \times \mathrm{Spin}_{10} \times \mathrm{SL}_{15}, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$ is prehomogeneous with $G_v^\circ \cong \mathrm{SL}_4$.

42. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_2, (\text{vecrep} \otimes \omega_1) \oplus (\text{halfspinrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{G}_2$.

43. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_3, (\text{halfspinrep} \otimes \omega_1) \oplus (\text{vecrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SL}_3 \times \mathrm{SO}_2$.

44. $(\mathrm{GL}_1^2 \times \mathrm{Spin}_{10} \times \mathrm{SL}_4, (\text{halfspinrep} \otimes \omega_1) \oplus (\text{vecrep} \otimes 1))$.
We have $G_v^\circ \cong \mathrm{SL}_2 \times \mathrm{SL}_2$.

45. $(\mathrm{GL}_1^2 \times \mathrm{G}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong \mathrm{SL}_2$.

46. $(\mathrm{GL}_1^2 \times \mathrm{G}_2 \times \mathrm{SL}_6, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong \mathrm{SL}_2$.

KI II Non-regular 2-simple prehomogeneous modules of type I.

1. (a) $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1} \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1^2 \cdot \mathrm{G}_1^+$.
(b) $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1} \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong (\mathrm{GL}_1 \times \mathrm{SO}_2) \cdot \mathrm{G}_{2n}^+$.
(c) $(\mathrm{GL}_1^2 \times \mathrm{SL}_{2n+1} \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes 3\omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{G}_n^+$.

2. (a) $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1} \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1^2 \cdot \mathrm{G}_{2n}^+$.
(b) $(\mathrm{GL}_1^3 \times \mathrm{SL}_{2n+1} \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{G}_{2n}^+$.

3. $(\mathrm{GL}_1^4 \times \mathrm{SL}_{2n+1} \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong \mathrm{GL}_1 \cdot \mathrm{G}_{2n}^+$.

4. $(GL_1^2 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong (GL_1 \times SO_2) \cdot Un_2$.
5. $(GL_1^3 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
6. $(GL_1^3 \times SL_4 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
7. (a) $(GL_1^2 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong (GL_1 \times SO_2) \cdot Un_2$.
(b) $(GL_1^2 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1))$.
We have $G_v^\circ \cong GL_1^2 \cdot Un_2$.
8. (a) $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
(b) $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1^2 \cdot Un_2$.
9. $(GL_1^3 \times SL_5 \times SL_9, (\omega_2 \otimes \omega_1) \oplus (\omega_1^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
10. $(GL_1^3 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes 2\omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
11. $(GL_1^4 \times SL_5 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
12. $(GL_1^2 \times SL_6 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^{(*)} \otimes 1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_3$.
13. (a) $(GL_1^2 \times SL_7 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_3$.
(b) $(GL_1^2 \times SL_7 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1))$.
We have $G_v^\circ \cong (GL_1 \times SO_2) \cdot Un_2$.
14. $(GL_1^3 \times SL_7 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1)$.
We have $G_v^\circ \cong GL_1 \cdot Un_2$.
15. $(GL_1^2 \times SL_9 \times SL_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1))$.
We have $G_v^\circ \cong GL_1 \cdot Un_3$.
16. (a) $(GL_1^2 \times Sp_n \times SL_{2m}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (GL_1 \times Sp_{m-1} \times Sp_{n-m-1}) \cdot Un_{2n-2}$.
(b) $(GL_1^2 \times Sp_n \times SL_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (GL_1 \times Sp_{m-1} \times Sp_{n-m-1}) \cdot Un_{2n-2}$.
17. $(GL_1^3 \times Sp_n \times SL_{2m}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (GL_1 \times Sp_{m-1} \times Sp_{n-m-1}) \cdot Un_{2n-2m-2}$.
18. (a) $(GL_1^4 \times Sp_n \times SL_{2m}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ for $n > m \geq 2$.
We have $G_v^\circ \cong (Sp_{m-2} \times Sp_{n-m-1}) \cdot Un_{2n-4}$.
(b) $(GL_1^4 \times Sp_n \times SL_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong Sp_{n-2} \cdot Un_{2n-3}$.

(c) $(GL_1^4 \times Sp_n \times SL_{2m}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ for $n > m \geq 2$.
We have $G_v^\circ \cong (Sp_{m-2} \times Sp_{n-m-1}) \cdot Un_{2n-4}$.

19. $(GL_1^3 \times Sp_n \times SL_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes 2\omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong Sp_{n-2} \cdot Un_{2n-3}$.

20. (a) $(GL_1^2 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (GL_1 \times Sp_m \times Sp_{n-m-1}) \cdot Un_{2n-2m-1}$.

(b) $(GL_1^2 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (GL_1^2 \times Sp_m \times Sp_{n-m-1}) \cdot Un_{2n-2m-3}$.

(c) $(GL_1^2 \times Sp_{m+1} \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$ for $m \geq 1$.
We have $G_v^\circ \cong (GL_1^2 \times Sp_{m-1}) \cdot Un_{4m-1}$.

(d) $(GL_1^2 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2))$ for $n > m+1 \geq 2$.
We have $G_v^\circ \cong (GL_1 \times SO_2^m \times Sp_{n-m-1}) \cdot Un_{2n-2m-1}$.

21. (a) $(GL_1^3 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1))$ for $n > m+1$.
We have $G_v^\circ \cong (Sp_{m-1} \times Sp_{n-m-2}) \cdot Un_{2n-4}$.

(b) $(GL_1^3 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$ for $n > m+1$.
We have $G_v^\circ \cong (Sp_{m-1} \times Sp_{n-m-2}) \cdot Un_{2n-4}$.

(c) $(GL_1^3 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)}))$ for $n > m+1$.
We have $G_v^\circ \cong (Sp_{m-1} \times Sp_{n-m-2}) \cdot Un_{2n-4}$.

22. (a) $(GL_1^4 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes (\omega_1 \oplus \omega_1)^{(*)}))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (Sp_{m-1} \times Sp_{n-m-1}) \cdot Un_{2n-2m-1}$.

(b) $(GL_1^4 \times Sp_n \times SL_{2m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$ for $n > m \geq 1$.
We have $G_v^\circ \cong (Sp_{m-2} \times Sp_{n-m-1}) \cdot Un_{2n+2m-7}$.

(c) $(GL_1^4 \times Sp_n \times SL_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$ for $n \geq 2$.
We have $G_v^\circ \cong Sp_{n-2} \cdot Un_{2n-3}$.

23. $(GL_1^2 \times Sp_n \times SL_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong (SO_2 \times Sp_{n-2}) \cdot Un_{2n-3}$.

24. $(GL_1^3 \times Sp_n \times SL_5, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*))$ for $n \geq 3$.
We have $G_v^\circ \cong (GL_1 \times Sp_{n-3}) \cdot Un_{2n-4}$.

25. $(GL_1^2 \times Sp_n \times SL_2, (\omega_1 \otimes 2\omega_1) \oplus (1 \otimes \omega_1))$ for $n \geq 2$.
We have $G_v^\circ \cong Sp_{n-2} \cdot Un_{2n-3}$.

26. $(GL_1^2 \times Spin_{10} \times SL_2, (\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
We have $G_v^\circ \cong (GL_1 \times G_2) \cdot G_1^+$.

2.4 2-Simple Prehomogeneous Modules of Type II

In this chapter we give a classification of the 2-simple prehomogeneous modules of type II, i.e. modules of the form

$$\Big(\mathrm{GL}_1^l \times G_1 \times G_2,\\ (\varrho_1 \otimes \tilde{\varrho}_1) \oplus \ldots \oplus (\varrho_k \otimes \tilde{\varrho}_k) \ \oplus\ (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \ \oplus\ (1 \otimes \tau_1) \oplus \ldots \oplus (1 \otimes \tau_t),\\ V_1 \oplus \ldots \oplus V_l\Big),$$

where all of the modules $(\mathrm{GL}_1 \times G_1 \times G_2, \varrho_i \otimes \tilde{\varrho}_i)$ are trivial prehomogeneous modules (see Kimura [3]). Note that we consider non-irreducible modules only. These were classified by Kimura et al. [6], thus we shall refer to them as KII n, where n is the number of the module in § 5 of [6]. Unfortunately, it is not always obvious from the classification in which cases a module would be prehomogeneous even with fewer than l scalar multiplications.

Any indecomposable 2-simple prehomogeneous module of type II is equivalent to one of the following:

KII I 2-simple prehomogeneous modules of type II obtained directly from any given simple module $(\mathrm{GL}_1^l \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$.

1. For any representation $\sigma_1 \oplus \ldots \oplus \sigma_s$ of G and $n \geq \sum_{i=1}^{s} \dim(\sigma_i)$:

$$\Big(\mathrm{GL}_1^{l+s} \times G \times \mathrm{SL}_n,\\ (\sigma_1 \otimes \omega_1) \oplus \ldots \oplus (\sigma_s \otimes \omega_1) \ \oplus\ (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1)\Big).$$

2. For $t \geq 0$, $1 \leq k \leq l$ and $n = t - 1 + \sum_{i=1}^{k} \dim(\varrho_i)$:

$$\Big(\mathrm{GL}_1^{l+t} \times G \times \mathrm{SL}_n,\\ (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \ \oplus\ (\varrho_{k+1}^* \otimes 1) \oplus \ldots \oplus (\varrho_l^* \otimes 1) \ \oplus\ (1 \otimes \omega_1^{\oplus t})\Big).$$

3. For $t \geq 1$, $1 \leq k \leq l$ and $n \geq t - 1 + \sum_{i=1}^{k} \dim(\varrho_i)$:

$$\Big(\mathrm{GL}_1^{l+t} \times G \times \mathrm{SL}_n,\\ (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \ \oplus\ (\varrho_{k+1} \otimes 1) \oplus \ldots \oplus (\varrho_l \otimes 1) \ \oplus\ (1 \otimes \omega_1^{\oplus t-1}) \oplus (1 \otimes \omega_1^*)\Big).$$

KII II 2-simple prehomogeneous modules of type II of the form

$$\Big(\mathrm{GL}_1^{k+s+t} \times G \times \mathrm{SL}_n,\\ (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \ \oplus\ (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \ \oplus\ (1 \otimes \tau_1) \oplus \ldots \oplus (1 \otimes \tau_t)\Big),$$

with $2 \leq \dim(\varrho_i) \leq n$ for all i and at least one $\tau_j \neq \omega_1^{(*)}$.

4. $G = \mathrm{SL}_m$ with $2 \leq m < n$.

4-i (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)})$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)}) \oplus (\omega_1^{(*)} \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)})$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1^{(*)} \otimes 1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1^{(*)} \otimes 1) \oplus (\omega_1^{(*)} \otimes 1)$.
(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (1 \otimes \omega_1^{(*)})$.
(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.

4-ii n even.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus ((\omega_1 \oplus \omega_1 \oplus \omega_1)^{(*)} \otimes 1)$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m even.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$, m even.
(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_2 \otimes 1)$, m odd.
(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (1 \otimes \omega_1^{(*)}) \oplus (1 \otimes \omega_1^{(*)})$, m odd.

4-iii n odd.
(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*)$, $m \geq 3$.
(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2^* \otimes 1)$.
(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus ((\omega_1 \oplus \omega_1 \oplus \omega_1)^{(*)} \otimes 1)$, $m \geq 3$.
(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1)$, m even.
(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1)$, m even.
(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)$, m even.
(h) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m even.
(i) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m odd.
(j) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, m odd.
(k) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^*)$, m odd.
(l) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
(m) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus ((\omega_1 \oplus \omega_1 \oplus \omega_1)^{(*)} \otimes 1)$.
(n) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.
(o) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m even.
(p) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m even.
(q) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$, m even.
(r) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_2 \otimes 1)$, m odd.
(s) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, m odd.
(t) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)$, m odd.

5. $G = SL_2$, $n > 2$.

5-i (a) $(2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)})$.

(b) $(2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1)$.

5-ii (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (3\omega_1 \otimes 1)$, n even.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$, n even.

(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*)$, n even.

5-iii (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (3\omega_1 \otimes 1)$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.

(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)})$.

5-iv $n = 5$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2^*)$.

5-v $n = 6$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (1 \otimes \omega_1^*)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1 \otimes 1)$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (2\omega_1 \otimes 1)$.

(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (3\omega_1 \otimes 1)$.

(f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.

(g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (2\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.

(h) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1) \oplus (\omega_1 \otimes 1)$.

5-vi $n = 7$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)})$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)}) \oplus (\omega_1 \otimes 1)$.

6. $G = SL_3$, $n > 3$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (2\omega_1^{(*)} \otimes 1)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2)$, $n = 5$.

7. $G = SL_4$, $n > 4$.

7-i n odd.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^{(*)} \otimes 1)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*)$.

7-ii $n = 5$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2)$.

7-iii $n = 6$.

(a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.

(b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^* \otimes 1)$.

(c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_2^{(*)} \otimes 1)$.

(d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
(e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (1 \otimes \omega_1)$.

8. $G = \mathrm{SL}_5$, $n > 5$.
 8-i n even.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^* \otimes 1)$.
 8-ii n odd.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^* \otimes 1)$.
 (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2) \oplus (\omega_2^* \otimes 1) \oplus (\omega_1 \otimes 1)$.
 (c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2) \oplus (\omega_2^* \otimes 1) \oplus (1 \otimes \omega_1^*)$.
 8-iii $n = 6$.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.
 (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^{(*)} \otimes 1)$.
 (c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_2^* \otimes 1)$.
 (d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1)$.
 (e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3) \oplus (1 \otimes \omega_1^*)$.
 8-iv $n = 7$.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)})$.
9. $G = \mathrm{SL}_{2j}$, $n = 2j + 1$.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$.
 (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3^{(*)}) \oplus (\omega_1^{(*)} \otimes 1)$, $j = 3$ (i.e. $n = 7$).
10. $G = \mathrm{SL}_n$.

$$(\omega_1 \otimes \omega_1) \;\oplus\; (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_k \otimes 1) \;\oplus\; (1 \otimes \varrho_{k+1}^*) \oplus \ldots \oplus (1 \otimes \varrho_r^*),$$

 where $(\mathrm{GL}_1^r \times \mathrm{SL}_n, \varrho_1 \oplus \ldots \oplus \varrho_r)$ is a simple prehomogeneous module.
11. $G = \mathrm{Sp}_m$, $2m < n$.
 11-i n odd.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2)$.
 (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1)$, $m = 2$.
 (c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_2 \otimes 1)$, $m = 2$.
 (d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*)$, $m = 2$.
 11-ii $n = 6$, $m = 2$.
 (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_3)$.

KII III 2-simple prehomogeneous modules of type II of the form

$$\Big(\mathrm{GL}_1^{k+s+t} \times G \times \mathrm{SL}_n,\; (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \;\oplus\; (\sigma_1 \otimes 1) \oplus \ldots \oplus (\sigma_s \otimes 1) \;\oplus\; (1 \otimes \omega_1^{\oplus t})\Big),$$

with $2 \le \dim(\varrho_i) \le n$ for all i and

$$(G,\ \varrho_1, \ldots, \varrho_k,\ \sigma_1, \ldots, \sigma_s) \neq (\mathrm{SL}_m,\ \omega_1, \ldots, \omega_1,\ \omega_1^{(*)}, \ldots, \omega_1^{(*)}).$$

12. $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_8, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.
13. $G = \mathrm{SL}_m$.
   13-i $m < n$.
   (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1)$.
   (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1) \oplus (\omega_1 \otimes 1)$.
   (c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^* \otimes 1) \oplus (\omega_1^* \otimes 1)$.
   (d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^*)$.
   (e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^* \otimes 1) \oplus (1 \otimes \omega_1)$.
   (f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2 \otimes 1) \oplus (\omega_1^* \otimes 1)$, m even.
   (g) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1)$, m even.
   (h) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_3 \otimes 1)$, $m = 6$.
   (i) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_3 \otimes 1) \oplus (1 \otimes \omega_1)$, $m = 6$.
   13-ii $n = m + 1$.
   (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2 \otimes 1)$.
   (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2^* \otimes 1)$, m even.
   (c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_3 \otimes 1)$, $m = 6$.
   13-iii $n \geq \frac{1}{2}m(m-1)$.
   (a) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m odd.
   (b) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, m odd, $n > \frac{1}{2}m(m-1)$.
   (c) $(\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, $m = 5$.
   (d) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $m = 2j+1$, $n = 2j^2 + j + 1$.
   (e) $(\omega_2 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $m = 2j$, $n = 2j^2 + j$.
   (f) $(\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $m = 5$, $n = 10$.
14. $G = \mathrm{Sp}_m$, $n \geq 2m$.
   (a) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$.
   (b) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_1 \otimes 1)$.
   (c) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$.
   (d) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1)$, $n > 2m$.
   (e) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (\omega_1 \otimes 1)$, $n = 2m$.
   (f) $(\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^{(*)})$, $n = 2m$.
15. $G = \mathrm{Spin}_{10}$, $n \geq 16$.
   (a) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$.
   (b) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*)$, $n \geq 17$.
   (c) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $n = 17$.
   (d) $(\text{halfspinrep} \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1)$, $n = 16$.

KII IV 2-simple prehomogeneous modules of type II of the form

$$\Big(\mathrm{GL}_1^{k+s_1+s_2+t_1+t_2} \times \mathrm{SL}_m \times \mathrm{SL}_n,$$
$$(\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \ \oplus\ (\omega_1 \otimes \omega_1)^{\oplus k} \ \oplus\ (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2}\Big),$$

where $n \geq m \geq 2$ and $k \geq 1$.

16. $n \geq km$.

  16-i $n = m$. Then $k = 1$ and $1 \leq (s_1 + t_2) + (s_2 + t_1) \leq n + 1$, where one of $s_1 + t_2$ or $s_2 + t_1$ is 0 or 1.

  16-ii $n = km, k \geq 2$.
  - (a) $t_1 = 0, 2 \leq t_2 \leq n, s_2 = 0, s_1 + kt_2 \leq m$.
  - (b) $t_2 = 0, 2 \leq t_1 \leq n, s_1 = 0, s_2 + kt_1 \leq m$.

  16-iii $n = km + 1$. Then $t_1 \geq 3, t_2 = s_1 = 0, s_2 + k(t_1 - 1) \leq m$.

  16-iv $n \geq km + t_1, n > km$.
  - (a) $k = 1, t_1 = 0, 2 \leq t_2 \leq n$ and $1 \leq (s_1 + t_2) + s_2 \leq m + 1$, where s_2 is 0 or 1.
  - (b) $k \geq 2, t_1 = 0, 2 \leq t_2 \leq n, s_2 = 0, s_1 + kt_2 \leq m$.
  - (c) $k \geq 1, t_1 = 1, 2 \leq t_2 \leq n, s_2 = 0, s_1 + kt_2 \leq m$.

17. $km > n$. These are the cases (17)-(25) in § 5.4 of [6], but to keep things simple we subsume them under the case KII IV-17 here. See the following definition 2.1 for the definition of T, $\nu(k, m, n)$ and (a_i). Also, we write $b_i = \frac{a_i}{a_{i+1}}$.

  17-i (a) $t_2 \geq 1, s_2 = t_1 = 0, s_1 + kt_2 \leq m - b_j(n - t_2)$, where $(k, m, n) \in T$ and $j = \nu(k, m, n)$.
  - (b) $s_2 = t_2 = 0$ and let $p = km + t_1 - n (< m), q = kp - m (< n), (k, p, m) \in T$ (resp. $(k, q, p) \in T$) and $j = \nu(k, p, m)$ (resp. $j = (k, q, p)$).
    - i. $m \geq kp, s_1 = 0$ and $t_1 \leq p + 1$.
    - ii. $m \geq kp, s_1 = 1$ and $k + t_1 \leq p + 1$.
    - iii. $m \geq kp, 2 \leq s_1 \leq m$ and $t_1 + ks_1 \leq p$.
    - iv. $kp > m, s_1 \geq 1$ and $t_1 + ks_1 \leq p - b_j(m - s_1)$.
    - v. $kp > m, p \geq kq, s_1 = 0, t_1 = 1$ and $k \leq q + 1$.
    - vi. $kp > m, p \geq kq, s_1 = 0, 2 \leq t_1 \leq p$ and $kt_1 \leq q$.
    - vii. $kp > m, kq > p, s_1 = 0, t_1 \geq 1$ and $kt_1 \leq q - b_j(p - t_1)$.
  - (c) $t_2 = 0, s_2 \geq 1, s_1 = 0$ and let $p = km + t_1 - n (< m), q = kp + s_2 - m (< p), r = kq - p (< q), (k, q, p) \in T$ (resp. $(k, q, p) \in T, (k, r, q) \in T$) and $j = \nu(k, q, p)$ (resp. $j = \nu(k, r, q)$).
    - i. $m \geq kp + s_2$ and $t_1 \leq p + 1$.
    - ii. $m = kp + s_2 - 1, t_1 = 0, 1$ and $k + t_1 \leq p + 1$.
    - iii. $m = kp + 1, m \geq s_2 \geq 3, t_1 = 0$ and $k(s_2 - 1) \leq p$.
    - iv. $m = kp, m \geq s_2 \geq 2$ and $ks_2 \leq p$.
    - v. $kp > m, p \geq kq, t_1 = 0$ and $s_2 \leq q + 1$.
    - vi. $kp > m, p \geq kq, t_1 = 1$ and $s_2 + kt_1 \leq q + 1$.
    - vii. $kp > m, p \geq kq, p \geq t_1 \geq 2$ and $s_2 + kt_1 \leq q$.
    - viii. $kp > m, kq > p, t_1 \geq 1$ and $s_2 + kt_1 \leq q - b_j(p - t_1)$.
    - ix. $kp > m, kq > p, q \geq kr, t_1 = 0, s_2 = 1$ and $k \leq r + 1$.
    - x. $kp > m, kq > p, q \geq kr, t_1 = 0, q \geq s_2 \geq 2$ and $ks_2 \leq r$.
    - xi. $kp > m, kq > p, kr > q, t_1 = 0, s_2 \geq 1$ and $ks_2 \leq r - b_j(q - s_2)$.

17-ii (a) $t_2 = 1, s_2 = 0, t_1 \geq 1$ and let $p = km + t_1 - n - 1$, $(k, p, m) \in T$ and $j = \nu(k, p, m)$.
  i. $m \geq kp$ and $(t_1 - 1) + k(k + s_1) \leq p$.
  ii. $kp > m$ and $(t_1 - 1) + k(k + s_1) \leq p - b_j(m - k - s_1)$.
(b) $t_2 = 0, s_2 \geq 1, s_1 = 1$ and let $p = km + t_1 - n$, $q = kp + s_2 - m - 1$, $(k, q, p) \in T$ and $j = \nu(k, q, p)$.
  i. $kp > m, p \geq kq, (s_2 - 1) + k(k + t_1) \leq q$.
  ii. $kp > m, kq > n, (s_2 - 1) + k(k + t_1) \leq q - b_j(p - k - t_1)$.

17-iii (a) $t_2 \geq 0, s_2 = 0, t_1 = 1, (s_1 + k) + k(t_2 - 1) \leq m - b_j(n - t_2)$ where $(k, m, n - 1) \in T$ and $j = \nu(k, m, n - 1)$.
(b) $t_2 = 0, s_2 = 1, s_1 \geq 2$ and let $p = km + t_1 - n (< m)$, $(k, p, m - 1) \in T$ and $j = \nu(k, p, m - 1)$.
  i. $m \geq kp$ and $t_1 + ks_1 \leq p$.
  ii. $kp > m$ and $t_1 + ks_1 \leq p - b_j(m - s_1)$.

17-iv (a) $t_2 = s_2 = 1$ and let $p = km + t_1 - n$, $(k, p, m - 1) \in T$ and $j = \nu(k, p, m - 1)$.
  i. $m - 1 \geq kp$ and $(k + t_1 - 2) + k(k + s_1 - 2) \leq p$.
  ii. $kp > m$ and $(k + t_1 - 2) + k(k + s_1 - 2) \leq p - b_j(n - k - s_1)$.

Definition 2.1 Let T be the set of triplets $(k, m, n) \in \mathbb{N}^3$ satisfying $k \geq 2$, $n > m \geq 2$ and $k + m^2 + n^2 - 2 > kmn$. For $(k, m, n) \in T$ there exists a $j \in \mathbb{N}$ such that $(GL_1^k \times SL_m \times SL_n, (\omega_1 \otimes \omega_1)^{\oplus k})$ is transformed to a trivial prehomogeneous module by j castling transformations. This number j is uniquely determined if we use only castling transformations decreasing the module's dimension. This unique j will be denoted by $\nu(k, m, n)$. Thus we obtain a map $\nu : T \to \mathbb{N}$. For example, $\nu(k, m, n) = 0$ if and only if $mk \leq n$. We define (a_i) to be the sequence

$$a_{-1} = -1, \qquad a_0 = 0, \qquad a_i = ka_{i-1} - a_{i-2} \text{ for } i > 0.$$

There are some cases of the form KII IV belonging neither to KII-16 nor KII-17, but to KII I instead. These are the cases (4.1-i), (4.1-ii), (4.7) and (4.8) from section 4.2 in Kimura et al. [6]. We will list them here for the sake of completeness.

1. $(GL_1^{1+s+t} \times SL_m \times SL_n, (\omega_1 \otimes 1)^{(*)\oplus s} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1})))$ with $k = 1$, $t \geq 1$, $n \geq m + t - 1$ and $s \leq m$. This is the case KII I-3.
2. $(GL_1^{k+s+t} \times SL_m \times SL_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^{(*)}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1})))$ with $k \geq 2$, $t \geq 1$, $n \geq km + t - 1$ and $s + k \leq m + 1$. This is the case KII I-3.
3. $(GL_1^{k+s+t} \times SL_m \times SL_n, (\omega_1^{(*)} \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$ with $n \geq km + t$, and $(GL_1^s \times SL_m, \omega_1^{(*)\oplus s})$ is a simple prehomogeneous module. This is the case KII I-1.
4. $(GL_1^{k+s+t} \times SL_m \times SL_n, (\omega_1^{(*)} \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$ with $t \geq 3$, $n = km + t - 1$, and $(GL_1^{k+s} \times SL_m, \omega_1^{*\oplus k} \oplus \omega_1^{(*)\oplus s})$ is a simple prehomogeneous module. This is the case KII I-2.

3 Tables of Étale Modules

In this chapter, we present the étale modules from part IV of Globke [1]. Note that this is not claimed to be a complete classification.

3.1 Étale Modules with Torus GL_1

3.1.1 1-Simple Étale Modules with Torus GL_1

- SK I-4: $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\Bbbk^2)$.
- Ks I-2: $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\Bbbk^n)^{\oplus n})$.

3.1.2 2-Simple Étale Modules with Torus GL_1

- SK I-8: $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2)$.
- SK I-11: $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^4)$.
- KII I-2: $(\mathrm{GL}_1 \times G \times \mathrm{SL}_n, (\varrho_1 \otimes \omega_1) \oplus \ldots \oplus (\varrho_k \otimes \omega_1) \oplus (\varrho_{k+1}^* \otimes 1) \oplus \ldots \oplus (\varrho_l^* \otimes 1))$, with $n = -1 + \sum_{i=1}^k \dim(\varrho_i)$ and $(\mathrm{GL}_1 \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ an étale module for a simple group G.
- KII IV-16-iv (a): $(\mathrm{GL}_m \times \mathrm{SL}_{m+1}, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m+1})$, $m \geq 2$.

3.2 Étale Modules with Sp_m

- KI I-16: $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- KI I-18: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-19: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.

3.3 All Étale Modules from Part IV of Globke [1]

3.3.1 1-Simple Étale Modules

- SK I-4: $(\mathrm{GL}_2, 3\omega_1, \mathrm{Sym}^3\Bbbk^2)$.
- Ks I-2: $(\mathrm{GL}_1 \times \mathrm{SL}_n, \mu \otimes \omega_1^{\oplus n}, (\Bbbk^n)^{\oplus n})$.
- Ks I-3: $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n+1}, (\Bbbk^n)^{\oplus n+1})$.
- Ks I-4: $(\mathrm{GL}_1^{n+1} \times \mathrm{SL}_n, \omega_1^{\oplus n} \oplus \omega_1^*, (\Bbbk^n)^{\oplus n} \oplus \Bbbk^{n*})$.
- Ks I-11 for $n = 2$: $(\mathrm{GL}_1^2 \times \mathrm{SL}_2, 2\omega_1 \oplus \omega_1, \mathrm{Sym}^2\Bbbk^2 \otimes \Bbbk^2)$.

3.3.2 2-Simple Étale Modules

- SK I-8: $(\mathrm{SL}_3 \times \mathrm{GL}_2, 2\omega_1 \otimes \omega_1, \mathrm{Sym}^2\Bbbk^3 \otimes \Bbbk^2)$.
- SK I-11: $(\mathrm{SL}_5 \times \mathrm{GL}_4, \omega_2 \otimes \omega_1, \bigwedge^2 \Bbbk^5 \otimes \Bbbk^4)$.
- KI I-1: $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1), (\bigwedge^2 \Bbbk^4 \otimes \Bbbk^2) \oplus (\Bbbk^4 \otimes \Bbbk^2))$.
- KI I-2: $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\bigwedge^2 \Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-6: $(\mathrm{GL}_1^3 \times \mathrm{SL}_5 \times \mathrm{SL}_2, (\omega_2 \otimes \omega_1) \oplus (\omega_1^* \otimes 1) \oplus (\omega_1^{(*)} \otimes 1), (\bigwedge^2 \Bbbk^5 \otimes \Bbbk^2) \oplus \Bbbk^{5*} \oplus \Bbbk^{5(*)})$.
- KI I-16: $(\mathrm{GL}_1^2 \times \mathrm{Sp}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^3) \oplus V^5 \oplus \Bbbk^3)$.
- KI I-18: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_2, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1), (\Bbbk^4 \otimes \Bbbk^2) \oplus \Bbbk^4 \oplus \Bbbk^2)$.
- KI I-19: $(\mathrm{GL}_1^3 \times \mathrm{Sp}_2 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^*), (\Bbbk^4 \otimes \Bbbk^4) \oplus \Bbbk^4 \oplus \Bbbk^4)$.
- KII I-1: $(\mathrm{GL}_1^j \times G \times \mathrm{GL}_n, ((\sigma_1 \oplus \ldots \oplus \sigma_s) \otimes \omega_1) \oplus ((\varrho_1 \oplus \ldots \oplus \varrho_l) \otimes 1))$, with $n = \sum_{i=1}^s \dim(\varrho_i)$ and $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ an étale module for a simple group G, $1 \leq j \leq l$.
- KII I-2: $(\mathrm{GL}_1^{j+t} \times G \times \mathrm{SL}_n, ((\varrho_1 \oplus \ldots \oplus \varrho_k) \otimes \omega_1) \oplus ((\varrho_{k+1}^* \oplus \ldots \oplus \varrho_l^*) \otimes 1) \oplus (1 \otimes \omega_1^{\oplus t}))$, with $n = t - 1 + \sum_{i=1}^k \dim(\varrho_i)$, $1 \leq j \leq l$, and $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ an étale module for a simple group G.
- KII I-3: $(\mathrm{GL}_1^{j+t} \times G \times \mathrm{SL}_n, ((\varrho_1 \oplus \ldots \oplus \varrho_k) \otimes \omega_1) \oplus ((\varrho_{k+1} \oplus \ldots \oplus \varrho_l) \otimes 1) \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, with $n = t - 1 + \sum_{i=1}^k \dim(\varrho_i)$, $1 \leq j \leq l$, and $(\mathrm{GL}_1^j \times G, \varrho_1 \oplus \ldots \oplus \varrho_l)$ an étale module for a simple group G.
- KII II-4-i (b): $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes 2\omega_1^{(*)}) \oplus (\omega_1^{(*)} \otimes 1))$.
- KII II-4-ii (a): $(\mathrm{GL}_1^5 \times \mathrm{SL}_3 \times \mathrm{SL}_4, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- KII II-4-iii (d): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1))$.
- KII II-4-iii (f): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1))$.
- KII II-4-iii (g): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*))$.
- KII II-4-iii (h): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$.
- KII II-4-iii (n): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus ((\omega_1 \oplus \omega_1)^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- KII II-4-iii (p): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*))$.
- KII II-4-iii (q): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (\omega_1 \otimes 1) \oplus (\omega_1^* \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- KII II-5-i (b): $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (2\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^{(*)}) \oplus (\omega_1 \otimes 1))$.
- KII II-5-ii (b): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1))$.

- KII II-5-iii (e): $(\mathrm{GL}_1^4 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (2\omega_1 \otimes 1) \oplus (1 \otimes \omega_1^{(*)}))$.
- KII II-5-iv (a): $(\mathrm{GL}_1^3 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_2^*))$.
- KII II-6 (b): $(\mathrm{GL}_1^3 \times \mathrm{SL}_3 \times \mathrm{SL}_5, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2) \oplus (1 \otimes \omega_2))$.
- KII II-9 (a): $(\mathrm{GL}_1^4 \times \mathrm{SL}_6 \times \mathrm{SL}_7, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_2^*) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1))$.
- KII II-10: $(\mathrm{GL}_1^{r+1} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (\varrho_1 \otimes 1) \oplus \ldots \oplus (\varrho_k \otimes 1) \oplus (1 \otimes \varrho_{k+1}^*) \oplus \ldots \oplus (1 \otimes \varrho_r^*))$, where $(\mathrm{GL}_1^r \times \mathrm{SL}_n, \varrho_1 \oplus \ldots \oplus \varrho_r)$ is an étale module.
- KII III-12: $(\mathrm{GL}_1^2 \times \mathrm{SL}_4 \times \mathrm{SL}_8, (\omega_2 \otimes \omega_1) \oplus (\omega_1 \otimes \omega_1))$.
- KII III-13-i (d): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^{(*)} \otimes 1) \oplus (1 \otimes \omega_1^*))$.
- KII III-13-i (e): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2^* \otimes 1) \oplus (1 \otimes \omega_1))$.
- KII III-13-i (g): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*) \oplus (1 \otimes \omega_1^*) \oplus (\omega_2 \otimes 1) \oplus (1 \otimes \omega_1))$.
- KII III-13-ii (a): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2 \otimes 1))$.
- KII III-13-ii (b): $(\mathrm{GL}_1^5 \times \mathrm{SL}_2 \times \mathrm{SL}_3, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\omega_2^* \otimes 1))$.
- KII IV-16-i: $(\mathrm{GL}_1^{n+2} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1)^{\oplus t_1} \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$,
  with $s_1 + t_2 = n$ and $s_2 + t_1 = 1$.
- KII IV-16-i: $(\mathrm{GL}_1^{n+2} \times \mathrm{SL}_n \times \mathrm{SL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$,
  with $s_1 + t_2 = n + 1$.
- KII IV-16-i: $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$,
  with $s_1 + t_2 = n$.
- KII IV-16-ii (a): $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t_2})$,
  with $s_1 + kt_2 = m$.
- KII IV-16-ii (b): $(\mathrm{GL}_n \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1)^{\oplus t_1})$,
  with $s_2 + kt_1 = m$.
- KII IV-16-iii: $(\mathrm{GL}_1^{t_1-1} \times \mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s_2} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1)^{\oplus t_1 - 1}$,
  with $s_2 + t_1 = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\mu \otimes \omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus m})$,
  with $n = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m})$,
  with $n = m + 1$.
- KII IV-16-iv (a): $(\mathrm{GL}_1^{m+2} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1^*)^{\oplus m+1})$,
  with $n = m + 1$.

- KII IV-16-iv (a): $(\mathrm{GL}_1^{m+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\mu \otimes \omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus m})$, with $n = m + 1$.

- KII IV-16-iv (c): $(\mathrm{GL}_1^{t_2} \times \mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s_1} \oplus (\omega_1 \otimes \omega_1) \oplus (1 \otimes \omega_1) \oplus (\mu \otimes 1 \otimes \omega_1^*)^{\oplus t_2})$, with $s_1 + t_2 = m$ and $n = m + 1$.

- KII IV-17 (1a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^*)^{\oplus t})$, with $sa_{j+1} + ta_{j+2} = m_0$ and $n_0 = km_0$.

Let $p = km + t - n$ and $q = kp - m = k^2m + kt - kn - m$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $s > 0$, $ta_{j+1} + sa_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_j m$, $\tilde{n}_0 = a_j p - a_{j-1} m$ and $j = \nu(k, p, m)$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq = p$, $2 \leq t$ and $kt = q$.

- KII IV-17 (1b): $(\mathrm{GL}_1^{k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $k = q + 1$ and $p = q^2 + q$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $k = q$ and $p = q^2$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq > p$, $ta_{j+2} = \tilde{m}_0$ and $k\tilde{m}_0 = \tilde{n}_0$ for $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1} p$ and $j = \nu(k, q, p)$.

- KII IV-17 (1b): $(\mathrm{GL}_1^{1+k+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp = m$ and $t + k = p + 1$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp = m$ and $t + k = p$.

- KII IV-17 (1b): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp = m$, $2 \leq s$ and $t + ks = p$.

Let $p = km + t - n$, $q = kp + s - m$ and $r = kq - p$.

- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq > p$, $t > 0$, $sa_{j+1} + ta_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1} p$ and $j = \nu(k, q, p)$.

- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $kr = q$, $2 \leq s$ and $ks = r$.

- KII IV-17 (1c): $(\mathrm{GL}_1^{k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $k = r + 1$ and $q = r^2 + r$.

- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $k = r$ and $q = r^2$.

- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $kp > m$, $kq > p$, $kr > q$, $sa_{j+2} = \tilde{m}_0$ and $k\tilde{m}_0 = \tilde{n}_0$ for $\tilde{m}_0 = a_{j+1}r - a_j q$, $\tilde{n}_0 = a_j r - a_{j-1}q$ and $j = \nu(k, r, q)$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{s+k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $kp > m$, $kq = p$ and $s + k = q + 1$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)$, with $kp > m$, $kq = p$ and $s + k = q$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1)^{\oplus t})$, with $kp > m$, $kq = p$, $2 \leq t$ and $s + kt = q$.
- KII IV-17 (1c): $(\mathrm{GL}_1^s \times \mathrm{SL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $k = p$ and $kp + s - 1 = m$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{s+k} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $k = p + 1$ and $kp + s - 1 = m$.
- KII IV-17 (1c): $(\mathrm{GL}_1^{s+k+1} \times \mathrm{SL}_m \times \mathrm{SL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1))$, with $k = p$ and $kp + s - 1 = m$.
- KII IV-17 (1c): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1^* \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k})$, with $ks = p$ and $kp = m$.

Let $p = km + t - n - 1$.

- KII IV-17 (2a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))$, with $kp > m$, $(t-2)a_{j+1} + (k+s)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_j m$, $\tilde{n}_0 = a_j p - a_{j-1}m$ and $j = \nu(k, p, m)$.
- KII IV-17 (2a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^* \oplus \omega_1^{\oplus t-1}))$, with $kp = m$ and $t - 2 + k(k+s) = p$.

Let $p = km + t - n$ and $q = kp + t - m - 1$.

- KII IV-17 (2b): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $kp > m$, $kq > p$, $(s-2)a_{j+1} + (k+t)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}q - a_j p$, $\tilde{n}_0 = a_j q - a_{j-1}p$ and $j = \nu(k, q, p)$.
- KII IV-17 (2b): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $kp > m$, $kq = p$ and $s - 2 + k(k+t) = q$.
- KII IV-17 (2b): $(\mathrm{GL}_1^{k+s+t} \times \mathrm{SL}_m \times \mathrm{SL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $m = s - 1 + kp$ and $k + t = p + 1$.
- KII IV-17 (2b): $(\mathrm{GL}_1^s \times \mathrm{SL}_m \times \mathrm{GL}_n, ((\omega_1 \oplus \omega_1^{*\oplus s-1}) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes \omega_1^{\oplus t})$, with $m = s - 1 + kp$ and $k + t = p$.
- KII IV-17 (3a): $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))$, with $(k+s)a_{j+1} + (t-2)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$, for $\tilde{m}_0 = a_{j+1}m - a_j(n-1)$, $\tilde{n}_0 = a_j m - a_{j-1}(n-1)$ and $j = \nu(k, m, n-1)$.

Let $p = km + t - n$.

- KII IV-17 (3b) $(\mathrm{GL}_m \times \mathrm{GL}_n, (\omega_1 \otimes 1)^{\oplus s} \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1 \oplus \omega_1^{*\oplus t-1}))$, with $kp > m$, $(k+t)a_{j+1} + (s-2)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$, for $\tilde{m}_0 = a_{j+1}p - a_j(m-1)$, $\tilde{n}_0 = a_j p - a_{j-1}(m-1)$ and $j = \nu(k, p, m-1)$.

Let $p = km + t - n - 1$.

- KII IV-17 (4): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, with $kp > m-1$, $(k+t-2)a_{j+1} + (k+s-2)a_{j+2} = \tilde{m}_0$ and $\tilde{n}_0 = k\tilde{m}_0$ for $\tilde{m}_0 = a_{j+1}p - a_j(m-1)$, $\tilde{n}_0 = a_j p - a_{j-1}(m-1)$ and $j = \nu(k, p, m-1)$.

- KII IV-17 (4): $(\mathrm{GL}_m \times \mathrm{GL}_n, ((\omega_1^{\oplus s-1} \oplus \omega_1^*) \otimes 1) \oplus (\omega_1 \otimes \omega_1)^{\oplus k} \oplus (1 \otimes (\omega_1^{\oplus t-1} \oplus \omega_1^*)))$, with $m - 1 = kp$ and $(k+t-2) + k(k+s-2) = p$.

4 Tables of Groups and Lie Algebras

4.1 The Classical Groups and their Lie Algebras

G	$\mathfrak{g} = \mathfrak{Lie}(G)$	$\dim_{\Bbbk}(\mathfrak{g})$
general linear group $\mathrm{GL}_n = \{A \in \mathrm{Mat}_n \mid \det(A) \neq 0\}$	$\mathfrak{gl}_n = \mathrm{Mat}_n$	n^2
special linear group $\mathrm{SL}_n = \{A \in \mathrm{Mat}_n \mid \det(A) = 1\}$	$\mathfrak{sl}_n = \{X \in \mathrm{Mat}_n \mid \mathrm{trace}(X) = 0\}$	$n^2 - 1$
orthogonal group $\mathrm{O}_n = \{A \in \mathrm{GL}_n \mid AA^\top = I_n\}$	$\mathfrak{o}_n = \{X \in \mathrm{Mat}_n \mid X^\top = -X\}$	$\frac{1}{2}n(n-1)$
special orthogonal group $\mathrm{SO}_n = \{A \in \mathrm{O}_n \mid \det(A) = 1\}$	$\mathfrak{so}_n = \mathfrak{o}_n$	$\frac{1}{2}n(n-1)$
unitary group $\mathrm{U}_n = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid AA^* = I_n\}$	$\mathfrak{u}_n = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \overline{X} = -X^\top\}$	n^2
special unitary group $\mathrm{SU}_n = \{A \in \mathrm{U}_n \mid \det(A) = 1\}$	$\mathfrak{su}_n = \{X \in \mathfrak{u}_n \mid \mathrm{trace}(X) = 0\}$	$n^2 - 1$
symplectic group $\mathrm{Sp}_n = \{A \in \mathrm{GL}_{2n} \mid A^\top JA = J\}$	$\mathfrak{sp}_n = \{X \in \mathrm{Mat}_{2n} \mid X^\top J + JX = 0\}$	$n(2n+1)$

Note that $\mathfrak{u}_n$ and $\mathfrak{su}_n$ are Lie algebras over $\Bbbk = \mathbb{R}$, but not over $\mathbb{C}$.

4.2 Complex Simple Lie-Algebras

Type	$\mathfrak{g}$		$\dim(\mathfrak{g})$
A_n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$n \geq 1$	$n^2 + 2n$
B_n	$\mathfrak{o}_{2n+1}(\mathbb{C})$	$n \geq 2$	$2n^2 + n$
C_n	$\mathfrak{sp}_n(\mathbb{C})$	$n \geq 3$	$2n^2 + n$
D_n	$\mathfrak{o}_{2n}(\mathbb{C})$	$n \geq 4$	$2n^2 - n$
G_2		-	14
F_4		-	52
E_6		-	72
E_7		-	133
E_8		-	248

Further we have $A_1 = B_1 = C_1$, $B_2 = C_2$ and $A_3 = D_3$.

4.3 Some Isomorphisms of Classical Lie Algebras

For an algebraically closed field $\mathbb{k}$ of characteristic 0, we have the following isomorphisms of Lie algebras:

$$\begin{aligned}
\mathfrak{so}_2(\mathbb{k}) &\cong \mathbb{k} \cong \mathfrak{gl}_1(\mathbb{k})\\
\mathfrak{sp}_1(\mathbb{k}) &\cong \mathfrak{sl}_2(\mathbb{k})\\
\mathfrak{so}_3(\mathbb{k}) &\cong \mathfrak{sl}_2(\mathbb{k})\\
\mathfrak{so}_4(\mathbb{k}) &\cong \mathfrak{sl}_2(\mathbb{k}) \oplus \mathfrak{sl}_2(\mathbb{k})\\
\mathfrak{so}_5(\mathbb{k}) &\cong \mathfrak{sp}_2(\mathbb{k})\\
\mathfrak{so}_6(\mathbb{k}) &\cong \mathfrak{sl}_4(\mathbb{k})
\end{aligned}$$

Not all of these isomorphisms hold over the real numbers. We have

$$\begin{aligned}
\mathfrak{sp}_1(\mathbb{R}) &\cong \mathfrak{sl}_2(\mathbb{R})\\
\mathfrak{so}_3(\mathbb{R}) &\cong \mathfrak{su}_2\\
\mathfrak{so}_4(\mathbb{R}) &\cong \mathfrak{su}_2 \oplus \mathfrak{su}_2\\
\mathfrak{so}_6(\mathbb{R}) &\cong \mathfrak{su}_4
\end{aligned}$$

Recall that two groups with isomorphic Lie algebras are locally isomorphic.

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