Crystallographic Groups II

Generalisations

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Classical Theory:
Study discrete cocompact (torsion-free) groups Γ of Euclidean isometries.

Generalisation:

- Study discrete groups of affine transformations;
  more specifically pseudo-Euclidean or symplectic ones.
- Find appropriate topological properties of their actions on $\mathbb{R}^n$.
- Consider groups with compact or non-compact quotients.

I. Discrete Groups and their Actions

(Proper) Discontinuity

Γ acts on a manifold X by homeomorphisms.
The action is called...

- free, if
  $\gamma.x = x$ for some $x \in X$ implies $\gamma = \mathrm{id}$.
- wandering (or discontinuous), if
  every $x \in X$ has a neighbourhood U_x such that the set

  $$\{\gamma \in \Gamma \mid \gamma.U_x \cap U_x \neq \emptyset\}$$

  is finite.
- properly discontinuous, if
  for every compact $K \subset X$ the set

  $$\{\gamma \in \Gamma \mid \gamma.K \cap K \neq \emptyset\}$$

  is finite.

Hierarchy of Properties

properly discontinuous
$\Rightarrow$ wandering
$\Rightarrow$ Γ is discrete (compact open topology)

Proper Definition of "Proper"?

Warning! Many authors...

- use the term "properly discontinuous" for what we call "wandering".
- assume that the action is also free
  (replace "is finite" by "$= \{\text{id}\}$").

Basic idea:

$$\Gamma \text{ wandering } \leftrightarrow \Gamma \text{ fundamental group}$$

$$\Gamma \text{ properly discontinuous} \leftrightarrow \Gamma \text{ fundamental group of Hausdorff space}$$

Characterisation of Proper Discontinuity

Theorem

Γ acts freely and properly discontinuously on a manifold X if and only if X/Γ is a manifold with fundamental group Γ.

Theorem

Γ acts properly discontinuously on a manifold X if and only if for all $x \in X$

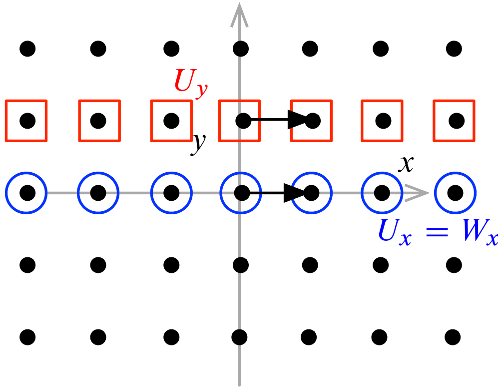
1. *Γ_x is finite,*
2. *there exists a Γ_x-invariant neighbourhood W_x of x such that $\gamma.W_x \cap W_x = \emptyset$ for all $\gamma \notin \Gamma_x$,*
3. *and for all $y \in X \setminus (\Gamma.x)$ there exist neighbourhoods U_x, U_y such that $\{\gamma \in \Gamma \mid \gamma.U_x \cap U_y \neq \emptyset\}$ is finite.*

Example 1: Properly Discontinuous Action

The group

$$\Gamma = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}$$

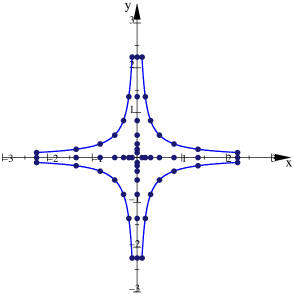
acts properly discontinuously by translations on $\mathbb{R}^2$.



Example 2: Non-Properly Discontinuous Action

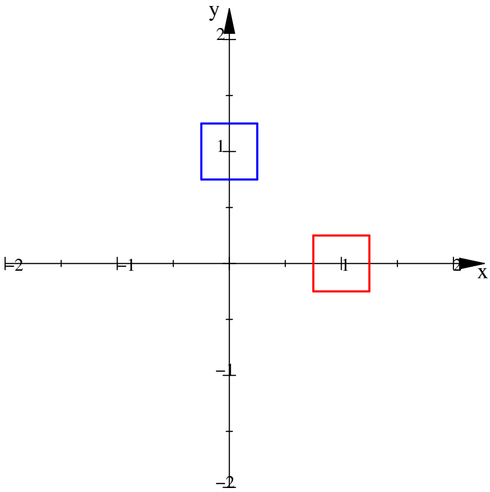
$\mathbb{Z}$ acts freely on $\binom{x}{y} \in \mathbb{R}^2 \setminus \{\binom{0}{0}\}$ by boosts:

$$\mathbb{Z} \to \mathbb{R}^2 \setminus \{\binom{0}{0}\}, \quad n \mapsto \begin{pmatrix} e^{\lambda n} & 0 \\ 0 & e^{-\lambda n} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



(the following figures use $\lambda = -\frac{1}{2}$)

Example 2: Non-Properly Discontinuous Action



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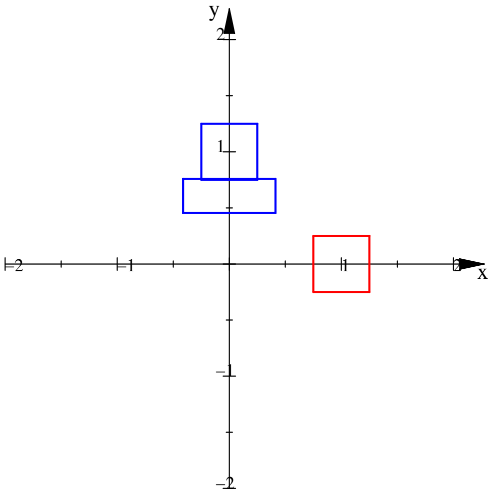

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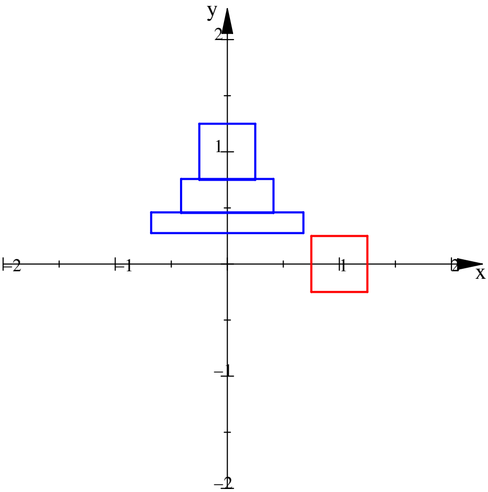

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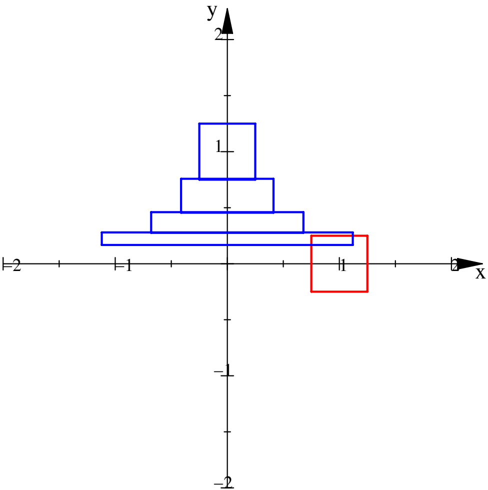
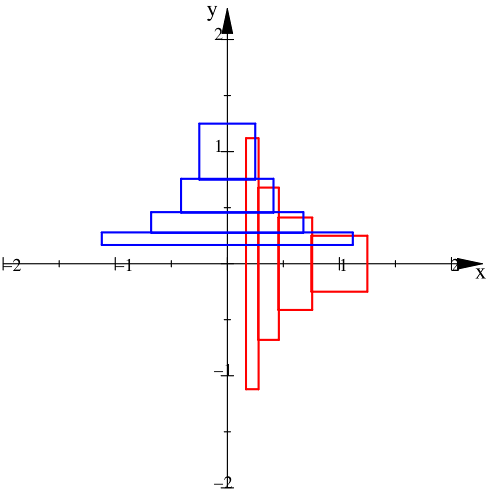


Example 2: Non-Properly Discontinuous Action

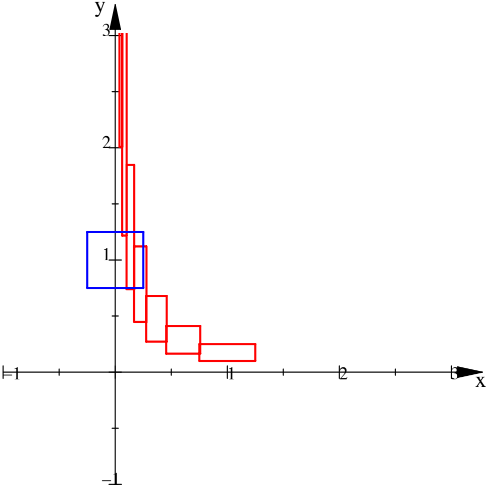


Example 3: Properly Discontinuous Action

Restrict the boost action of $\mathbb{Z}$ to $\mathbb{R}^2 \setminus \{x\text{-axis}\}$:
The action becomes properly discontinuous!

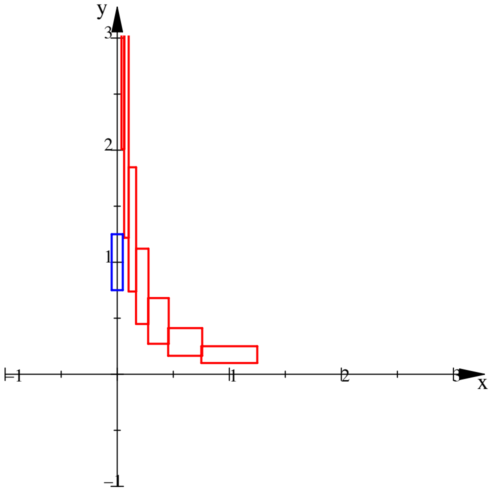
Example 3: Properly Discontinuous Action

Only finitely many intersections. . .



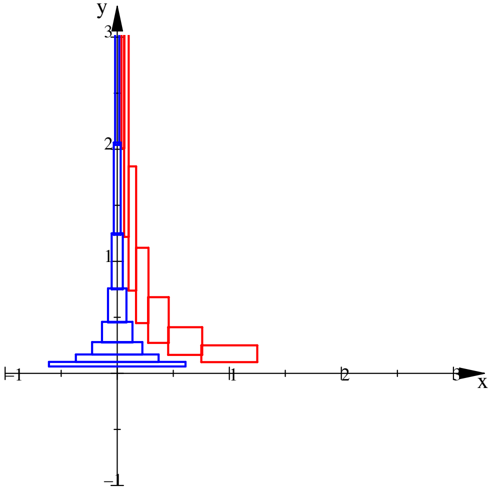
Example 3: Properly Discontinuous Action

. . . pick smaller neighbourhood:



Example 3: Properly Discontinuous Action

No more intersections.



Proper Discontinuity on Riemannian Manifolds

Fact

Let M be a Riemannian manifold with isometry group $\mathbf{Iso}(M)$. Every discrete subgroup $\Gamma \subset \mathbf{Iso}(M)$ acts properly discontinuous.

Recall Bieberbach groups:

- $\Gamma \subset \mathbf{Iso}(\mathbb{R}^n)$ discrete $\Leftrightarrow$ Γ properly discontinuous
- Γ torsion-free $\Leftrightarrow$ Γ-action free

This does not generalise to pseudo-Riemannian isometry groups!

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II. Flat Affine Manifolds

Affine Crystallographic Groups

A group $\Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$ is called an affine crystallographic group if the action of Γ on $\mathbb{R}^n$ is free and properly discontinuous with compact quotient.

A manifold M with a torsion-free affine connection ∇ is called an affine manifold.

Affine Killing-Hopf Theorem

Let M be a geodesically complete flat affine manifold. Then M is affinely equivalent to $\mathbb{R}^n/\Gamma$, where the Γ is the fundamental group of M (in particular, Γ acts freely and properly discontinuously).

Equivalence

Identify affinely equivalent groups:

$$\Gamma_1 \sim \Gamma_2 \quad :\Leftrightarrow \quad \Gamma_1 = g \cdot \Gamma_2 \cdot g^{-1} \text{ for some } g \in \textbf{Aff}(\mathbb{R}^n)$$

Do Bieberbach's theorems generalise to classes of affine crystallographic groups?

Bieberbach's First Theorem?

Bieberbach's First Theorem does not hold:

- $\Gamma \cap \mathbb{R}^n$ does not necessarily span $\mathbb{R}^n$.
- $\mathrm{L}(\Gamma)$ is not necessarily finite.

Example

The group

$$\Gamma = \left\{ \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & a & 1 & c \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \;\middle|\; a, b, c \in \mathbb{Z} \right\} \subset \mathbf{Aff}(\mathbb{R}^3)$$

is an affine crystallographic group acting on $\mathbb{R}^3$.
Clearly,

- $\Gamma \cap \mathbb{R}^n$ spans only a 2-dimensional subspace.
- $\mathrm{L}(\Gamma) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{Z} \right\}$ is not finite.

Auslander's Conjecture

A tentative analogue to Bieberbach's First Theorem is

Conjecture (Auslander, 1964)
If $\Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$ is an affine crystallographic group, then Γ is virtually polycyclic.

Here, a group Γ is called...

- polycyclic if there exists a sequence of subgroups

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_k = \mathbf{1}$$

  such that all Γ_j/Γ_{j+1} are cyclic groups.
- virtually polycyclic if Γ contains a polycyclic subgroup Γ' of finite index (also: polycyclic-by-finite).

Auslander's Conjecture

Auslander's Conjecture has been proven in special cases:

- $\Gamma \subset \textbf{Aff}(\mathbb{R}^3)$ (Fried & Goldman, 1983)
- $\Gamma \subset \textbf{Iso}(\mathbb{R}^n_1)$ (Lorentz metric)
  - Conjecture holds for complete compact flat Lorentz manifolds (Goldman & Kamishima, 1984)
  - Compact flat Lorentz manifolds are complete (Carriere, 1989)
  - Classification is known (Grunewald & Margulis, 1989)

Milnor's Conjecture

Milnor dropped Auslander's restriction that Γ acts cocompactly.

Theorem (Milnor, 1977)

Let Γ be a torsion-free and virtually polycyclic group.
Then Γ is isomorphic to the fundamental group of some complete flat affine manifold.

Conjecture (Milnor, 1977)

The fundamental group of a flat affine manifold is virtually polycyclic.

Margulis Spacetime

Milnor's conjecture is wrong!

- Discrete subgroups $\mathbb{Z} * \mathbb{Z} \subset \mathbf{O}_{2,1}$ are known.
- Augment $\mathbb{Z} * \mathbb{Z}$ by translation parts so that the action on $\mathbb{R}^3_1$ is properly discontinuous (Margulis, 1983).
- Note: These Margulis spacetimes are not compact, so Auslander's conjecture is still open.

Bieberbach's Second Theorem?

$\Gamma_1 \cong \Gamma_2$ does not necessarily imply $\Gamma_1 \sim \Gamma_2$.

Example

The affine crystallographic group Γ_1, Γ_2 are both isomorphic to $\mathbb{Z}^3$:

$$\Gamma_1 = \left\langle \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \right\rangle,$$

$$\Gamma_2 = \left\langle \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \right\rangle.$$

But Γ_2 has trivial holonomy, Γ_1 does not.

Bieberbach's Third Theorem?

There are infinitely many affine equivalence classes of affine crystallographic groups.

Example

For fixed $k \in \mathbb{Z}$ define an affine crystallographic group

$$\Gamma_k = \left\{ \left(\begin{array}{ccc|c} 1 & 0 & 0 & ka \\ 0 & 1 & 0 & kb \\ 0 & ka & 1 & kc \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \;\middle|\; a, b, c \in \mathbb{Z} \right\} \subset \mathbf{Aff}(\mathbb{R}^3).$$

Then for $m \neq n$,

$$\Gamma_m \ncong \Gamma_n.$$

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III. Homogeneous Flat Affine Manifolds

Homogeneous Flat Manifolds

A more tractable class of spaces are the homogeneous flat affine (or pseudo-Riemannian) manifolds; those with a transitive group of affinities (or isometries).

Theorem

Let M be a flat affine manifold with fundamental group Γ. Then M is homogeneous if and only the centraliser $Z_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma)$ of Γ in $\mathbf{Aff}(\mathbb{R}^n)$ acts transitively.

Proof:

- $\mathbf{Aff}(M) = \mathsf{N}_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma)/\Gamma$ (normaliser).
- Γ is discrete, so $\mathsf{Z}_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma) \supseteq \mathsf{N}_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma)^\circ$.
- M homogeneous if and only if $\mathbf{Aff}(M)^\circ$ acts transitively. □

Unipotent Groups

A matrix group G is called unipotent if there exists $k \in \mathbb{N}$ such that all $g \in G$ satisfy

$$(\mathrm{I}_n - g)^k = 0.$$

A unipotent group is a nilpotent group.

Example:

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Fundamental Groups of Homogeneous Flat Spaces

Theorem

The fundamental group Γ of a complete homogeneous flat affine manifold M is unipotent (in particular, Γ is nilpotent).

Proof:

- As $Z_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma)$ acts transitively, $G = Z_{\mathbf{Aff}(\mathbb{R}^n)}(Z_{\mathbf{Aff}(\mathbb{R}^n)}(\Gamma))$ acts freely.
- G is an algebraic subgroup of $\mathbf{Aff}(\mathbb{R}^n)$, so it has Chevalley decomposition $G = R \cdot U$ with R reductive, U unipotent.
- But an affine reductive algebraic group R has a fixed point on $\mathbb{R}^n$, so by the first step: $G = U$ is unipotent.
- Clearly, $\Gamma \subset G$ is also unipotent. □

Fact (Fried, Goldman & Hirsch, 1981)

If M is complete, compact and Γ is nilpotent, then M is homogeneous.

Flat Pseudo-Riemannian Homogeneous Manifolds

Theorem (Wolf, 1962)

Let Γ be the fundamental group of a flat pseudo-Riemannian homogeneous manifold M. Then:

- Γ *is 2-step nilpotent (meaning* $[\Gamma, [\Gamma, \Gamma]] = \{id\}$*).*
- *Every* $\gamma \in \Gamma$ *is of the form* $\gamma = (\mathrm{I}_n + A, v)$ *with* $A^2 = 0$ *and* $Av = 0$.
- *The image of* A *is totally isotropic and orthogonal to* v.

Example

Wolf assumed all Γ were in fact abelian.

Example (Baues, 2008)
Let $G = \mathrm{H}_3 \ltimes_{\mathrm{Ad}^*} \mathfrak{h}_3^*$ and Γ a lattice in G,
with bi-invariant inner product of signature $(3,3)$ defined by

$$\langle (X, \xi), (Y, \eta) \rangle = \xi(Y) + \eta(X),$$

$X, Y \in \mathfrak{h}_3$, $\xi, \eta \in \mathfrak{h}_3^*$.

Then

$$M = G/\Gamma$$

is a compact flat pseudo-Riemannian manifold with transitive G-action and non-abelian fundamental group.
However, $\mathbf{Hol}(M) = \mathrm{L}(\Gamma)$ is abelian.

Compactness

Theorem (Baues, 2008)

If M is a compact flat pseudo-Riemannian homogeneous manifold, then $\mathbf{Hol}(M)$ is abelian.

Holonomy

Theorem

With respect to a certain Witt basis of $\mathbb{R}^n$*, the holonomy group* $\mathrm{L}(\Gamma)$ *of a flat pseudo-Riemannian homogeneous manifold takes the form*

$$\mathrm{L}(\gamma) = \begin{pmatrix} \mathrm{I}_k & -B^\top \tilde{\mathrm{I}} & C \\ 0 & \mathrm{I}_{n-2k} & B \\ 0 & 0 & \mathrm{I}_k \end{pmatrix},$$

where $C \in \mathfrak{so}_k$*, and* $-B^\top \tilde{\mathrm{I}} B = 0$*, where* $\tilde{\mathrm{I}}$ *defines a non-degenerate bilinear form on a certain subspace of* $\mathbb{R}^n$*.*

$\mathrm{L}(\Gamma)$ *is abelian if and only if* $B = 0$ *for all* $\gamma \in \Gamma$*.*

Non-Abelian Holonomy

Theorem

Let M be a flat pseudo-Riemannian homogeneous manifold.
If $\mathbf{Hol}(M)$ is not abelian, then

$$\dim M \geq 8.$$

If in addition M is complete, then

$$\dim M \geq 14.$$

Examples show that both bounds are sharp.

Realisations as Fundamental Groups

Theorem

Let Γ be a finitely generated torsion-free 2-step nilpotent group of rank n.
Then there exists a complete flat pseudo-Riemannian homogeneous manifold M with fundamental group Γ, and $\dim M = 2n$.

Proof:

- Let H be the Malcev hull of Γ (an algebraic group such that Γ embeds as a lattice in H, and $\dim H = n$).
- Set $G = H \ltimes_{\mathrm{Ad}^*} \mathfrak{h}^*$ and define a flat bi-invariant inner product by $\langle (X,\xi),(Y,\eta)\rangle = \xi(Y) + \eta(X)$.
- The action of $\gamma \in \Gamma$ on G by $\gamma.(h,\xi) = (\gamma h, \mathrm{Ad}^*(\gamma)\xi)$ is isometric.
- So $M = G/\Gamma$ is a flat pseudo-Riemannian homogeneous manifold. □

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